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Application of iteration perturbation method and Hamiltonian approach for nonlinear vibration of Euler-Bernoulli beams

Abstract
This paper is devoted to the new classes of analytical techniques called the Iteration Perturbation Method (IPM) and Hamiltonian Approach (HA) for solving the equation of motion governing the nonlinear vibration of Euler-Bernoulli beams subjected to the axial loads. It has been found that the IPM and HA are very prolific, rapid, functional and do not demand small perturbation and are also sufficiently accurate to both linear and nonlinear problems in engineering. Comparison of the results of these methods with together and with the results of numerical solution reveals that the IPM and HA are very effective and convenient, and can be easily extended to other nonlinear systems so that can be found widely applicable in engineering and other sciences.

Keywords:
Nonlinear vibration, Euler-Bernoulli beam, Iteration Perturbation Method, Hamiltonian Approach.

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1 INTRODUCTION

Vibration analysis of the beams is an important issue in structural engineering applications such as long span bridges, aerospace vehicles, automobiles and many other industrial usages. The dynamics of continuous or distributed parameter systems, such as beams, plates, and shells, are governed by nonlinear partial-differential equations in space and time. These partial differential equations and associated boundary conditions form an initial boundary-value problem (Bayat et al., 2011a).

In general it is hard to find exact or closed-form solutions for this class of problems. Consequently, approximate solutions of the original problem were sought. The study of nonlinear vibration equations solution has been applied by many researchers and various methods of solution have been used (Bayat et al., 2011b). In recent years, much attention has been devoted to the newly developed methods to construct an analytic solution for nonlinear vibration such as He's Homotopy Perturbation Method (HPM) (He, 2012; He, 2004; Bayat et al., 2012; Ganji, 2006; Beléndez et al., 2007; Barari et al., 2008), Homotopy Analysis Method (HAM) (Bayat et al., 2012; Liao, 2003; Ghotbi et al., 2009; Sohouli et al., 2010; Kimiaeifar et al., 2009), He's Parameter-Expansion Method (He, 2012; Bayat et al., 2012; He and Shou, 2007; Xu, 2007; Kimiaeifar et al., 2010a; Kimiaeifar et al., 2010b), He's Variational Iteration Method (VIM) (He, 2012; Bayat et al., 2012; He, 1997; He, 2000; Ganji et al., 2007; Barari et al., 2011; Rafei et al., 2007), He's Energy Balance Method (EBM) (He, 2002; Bayat et al., 2012; Jamshidi and Ganji, 2010; Ganji et al., 2009; Fu et al., 2011; Afrouzi et al., 2011; Sfahani et al., 2011; Momeni et al., 2011; Zhang et al., 2009a), He's Amplitude Frequency Formulation Method (HAFF) (He, 2012; Bayat et al., 2012; He, 2008; Ganji et al., 2010; Ren et al., 2009; Zhang, 2009b), Iteration Perturbation Method (IPM) (Bayat et al., 2012; He, 2001; Ozis and Yildirim, 2009; Bayat et al., 2011b), Hamiltonian Approach (HA) (He, 2012; Bayat et al., 2012; Bayat and Pakar, 2011c; Khan et al., 2010; Xu, 2010; Yazdi et al., 2010; Yildirim et al., 2011a; Yildirim et al., 2011b), Exp–Function Method (He, 2012; Wu and He, 2007; Davodi et al., 2010), Adomian's Decomposition Method (Sadighi and Ganji, 2007; Safari et al., 2009; Biazar et al., 2010), Differential Transformation Method (DTM) (Borhanifar and Abazari, 2010; Ghafoori et al., 2011; Joneidi et al., 2009), Variational Approach (VA) (He, 2007; He, 2012; Bayat et al., 2012; Amani et al., 2011; Ganji et al., 2008; Shou, 2009; Zhang, 2007), Harmonic Balance Method (HBM) (Beléndez et al., 2009), Parameterized Perturbation Method (PPM) (He, 1999; He, 2006; He, 2012; Bayat et al., 2012; Ding and Zhang, 2009; Jalaal, et al., 2011).

Among these methods, Iteration Perturbation Method (IPM) and Hamiltonian Approach (HA) are considered to solve the nonlinear vibration of Euler-Bernoulli beams subjected to the axial loads in this paper. The paper has been collocated as follows: first, the mathematical formulation of the problem is considered, and then the basic concept of IPM and HA is described. Subsequently, IPM and HA are studied to demonstrate the applicability and preciseness of these methods; this is followed by a presentation of some comparisons between analytical and numerical solutions. Eventually, it is showed that IPM and HA can converge to a precise cyclic solution for nonlinear systems.

2 Mathematical Formulation

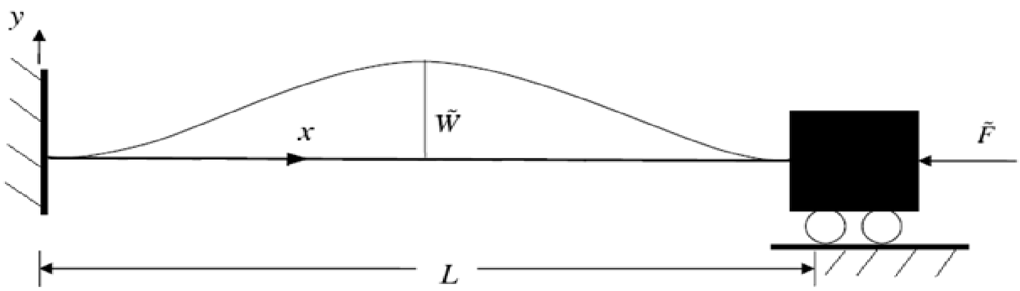

Figure 1 A schematic of an Euler-Bernoulli beam subjected to an axial load.

Consider a straight beam of length L, a cross-section A, a mass per unit length μ, moment of inertia I , and modulus of elasticity E that subjected to an axial force of magnitude $\tilde{F}$, as shown in Fig. 1. The equation of motion including of effects of mid-plane stretching is given by (Lestari and Hanagud; 2001; Lacarbonara, 1997):

$$EI\frac{\partial^4\tilde{W}}{\partial\tilde{X}^4}+\mu\frac{\partial^2\tilde{W}}{\partial\tilde{t}^2}+\tilde{F}\frac{(\partial^2\tilde{W})}{\partial\tilde{X}^2}+C\frac{\partial\tilde{W}}{\partial\tilde{t}}-\frac{EA}{2L}\frac{\partial^2\tilde{W}}{\partial\tilde{X}^2}\int_0^L\left(\frac{\partial\tilde{W}}{\partial\tilde{X}}\right)^2d\tilde{X}=U(\tilde{X},t) \tag{1}$$

Where C is the viscous damping coefficient, and U is a distributed load in the transverse direction.
Assume the non-conservative forces were equal to zero. Therefore, Eq. (1) can be written as follows:

$$EI\frac{\partial^4\tilde{W}}{\partial\tilde{X}^4}+\mu\frac{\partial^2\tilde{W}}{\partial\tilde{t}^2}+\tilde{F}\frac{(\partial^2\tilde{W})}{\partial\tilde{X}^2}-\frac{EA}{2L}\frac{\partial^2\tilde{W}}{\partial\tilde{X}^2}\int_0^L\left(\frac{\partial\tilde{W}}{\partial\tilde{X}}\right)^2d\tilde{X}=0 \tag{2}$$

For convenience, the following non-dimensional variables are use:

$$X=\frac{\tilde{x}}{L}\qquad W=\frac{\tilde{w}}{R},\quad t=\tilde{t}\sqrt{\frac{EI}{ML^4}},\quad F=\frac{\tilde{F}L^2}{EI} \tag{3}$$

Where $R=\sqrt{\frac{I}{A}}$ is the radius of gyration of the cross-section. As a result, Eq. (2) can be written as follows:

$$\frac{\partial^4W}{\partial X^4}+\frac{\partial^2W}{\partial t^2}+F\frac{\partial^2W}{\partial X^2}-\frac{1}{2}\frac{\partial^2W}{\partial X^2}\int_0^L\left(\frac{\partial W}{\partial X}\right)^2dx=0 \tag{4}$$

Assuming $W(X,t)=\phi(X)Q(t)$ where $Q(t)$ is the first eigenmode of the beam (Tse et al., 1978) and applying the Galerkin method, the equation of motion is obtained as follows:

$$\ddot{Q}(t)+\alpha\,Q(t)+\beta\,Q^3(t)=0 \tag{5}$$

Where $\alpha=\alpha_1+F\alpha_2$ and α_1,α_2 and β are as follows:

$$\alpha_1=\frac{\int_0^1\phi^{(iv)}\phi\,dx}{\int_0^1\phi^2\,dx},\quad \alpha_2=\frac{\int_0^1\phi''\phi\,dx}{\int_0^1\phi^2\,dx},\quad \beta=\frac{-\frac{1}{2}\int_0^1\left(\phi''\int_0^1\phi''^2\,dx\right)\phi\,dx}{\int_0^1\phi^2\,dx} \tag{6}$$

The Eq. (5) is the differential equation of motion governing non-linear vibration of Euler-Bernoulli beams. The center of the beam is subjected to the following initial conditions:

$$Q(0)=A,\qquad \dot{Q}(0)=0 \tag{7}$$

Where Q_{max} denotes the non-dimensional maximum amplitude of oscillation.

3. Solution Procedures

3.1 Basic concept of Iteration Perturbation Method (IPM)

The Iteration Perturbation Method was first proposed by He (2001). To illustrate its basic solution process, we consider the following general nonlinear oscillator:

$$\ddot{u}+f\left(u,t\right)=0 \tag{8}$$

The variable $y=\frac{du}{dt}$ is introduced, and then Eq. (8) can be replaced by equivalent system

$$\dot{u}\left(t\right)=y\left(t\right), \tag{9}$$

$$\dot{y}\left(t\right)=-f\left(u,t\right). \tag{10}$$

With assumption that its initial approximate can be expressed as

$$u\left(t\right)=A\cos\left(\omega t\right) \tag{11}$$

Where ω is the angular frequency of the oscillation. Then:

$$\dot{u}\left(t\right)=A\sin\left(\omega t\right)=y\left(t\right) \tag{12}$$

Substituting Eq. (11) and Eq. (12) into the Eq. (10) yields:

$$\dot{y}\left(t\right)=-f\left(A\cos\left(\omega t\right),t\right) \tag{13}$$

Using Fourier expansion series in the right hand of Eq. (13):

$$f\left(A\cos\left(\omega t\right),t\right)=\sum_{n=0}^{\infty}\alpha_{2n+1}\cos\left[\left(2n+1\right)\omega t\right]=\alpha_1\cos\left(\omega t\right)+\alpha_3\cos\left(3\omega t\right)+\ ... \tag{14}$$

Substituting Eq. (14) into Eq. (13) yields:

$$\dot{y}\left(t\right)=-\left(\alpha_1\cos\left(\omega t\right)+\alpha_3\cos\left(3\omega t\right)+\ ...\right) \tag{15}$$

Integrating Eq. (15) yields:

$$y\left(t\right)=-\frac{\alpha_1}{\omega}\sin\left(\omega t\right)-\frac{\alpha_3}{3\omega}\sin\left(3\omega t\right)-\ ... \tag{16}$$

Comparing Eq. (12) and (16):

$$-A\omega = -\frac{\alpha_1}{\omega}, \tag{17}$$

$$\omega = \sqrt{\frac{\alpha_1}{A}}. \tag{18}$$

3.2 Basic concept of Hamiltonian Approach (HA)

Previously, He (2002) had introduced the energy balance method based on collocation and Hamiltonian. Recently, in 2010 it was developed into the Hamiltonian approach (He, 2010). This approach is a kind of energy method with a vast application in conservative oscillatory systems. In order to clarify this approach, the following general oscillator is considered (He, 2010):

$$\ddot{u} + f\left(u, \dot{u}, \ddot{u}\right) = 0 \tag{19}$$

With initial conditions:

$$u\left(0\right) = A, \quad \dot{u}\left(0\right) = 0 \tag{20}$$

Oscillatory systems contain two important physical parameters, i.e. the frequency ω and the amplitude of oscillation A. It is easy to establish a variational principle for Eq. (19), which reads:

$$J\left(u\right) = \int_0^{\frac{T}{4}} \left(-\frac{1}{2}\dot{u}^2 + F\left(u\right)\right) dt \tag{21}$$

Where T is period of the nonlinear oscillator $\frac{\partial F}{\partial u} = f$.

In the Eq. (21), $\frac{1}{2}\dot{u}^2$ is kinetic energy and $F(u)$ potential energy, so the Eq. (21) is the least Lagrangian action, from which we can immediately obtain its Hamiltonian, which reads

$$\mathrm{H}\left(u\right) = \frac{1}{2}\dot{u}^2 + F\left(u\right) = \text{constant} \tag{22}$$

From Eq. (22):

$$\frac{\partial H}{\partial A} = 0 \tag{23}$$

Introducing a new function, $\bar{H}(u)$, defined as:

$$\bar{\mathrm{H}}\left(u\right) = \int_0^{\frac{T}{4}} \left(\frac{1}{2}\dot{u}^2 + F\left(u\right)\right) \mathrm{dt} = \frac{1}{4}TH \tag{24}$$

Eq. (23) is, then, equivalent to the following one

$$\frac{\partial}{\partial A}\left(\frac{\partial \bar{H}}{\partial T}\right)=0 \tag{25}$$

Or

$$\frac{\partial}{\partial A}\left(\frac{\partial \bar{H}}{\partial (1/\omega)}\right)=0 \tag{26}$$

From Eq. (26) approximate frequency–amplitude relationship of a nonlinear oscillator can be obtained.

4 Runge-Kutta

The fourth RK (Runge-Kutta) method has been used for the numerical approach to verify the analytic solution. This iterative algorithm was written in the form of the following formulae for the second-order differential equation:

$$\begin{aligned} \dot{x}_{i+1} &= \dot{x}_i + \frac{\Delta t}{6}\left(h_1 + 2h_2 + 2h_3 + k_4\right) \\ x_{i+1} &= x_i + \Delta t\left(\dot{x}_i + \frac{\Delta t}{6}\left(h_1 + h_2 + k_3\right)\right) \end{aligned} \tag{27}$$

Where Δt is the increment of the time and h_1, h_2, h_3 *and* h_4 and 4 h are determined from the following formulae:

$$\begin{aligned} h_1 &= f\,\left(\dot{x}, x_i, \dot{x}_i\right)k, \\ h_2 &= f\left(t_i + \frac{\Delta t}{2}, x_i + \frac{\Delta t}{2}\dot{x}_i, \dot{x}_i + \frac{\Delta t}{2}h_1\right)k, \\ h_3 &= f\left(t_i + \frac{\Delta t}{2}, x_i + \frac{\Delta t}{2}\dot{x}_i, \frac{1}{4}\Delta t^2 h_1, \dot{x}_i + \frac{\Delta t}{2}h_2\right)k, \\ h_4 &= f\left(t_i + \Delta t, x_i + \Delta t\,\dot{x}_i, \frac{1}{2}\Delta t^2 h_2, \dot{x}_i + \Delta t\,\mathrm{h}_3\right)k. \end{aligned} \tag{28}$$

The numerical solution starts from the boundary at the initial time, where the first value of the displacement function and its first-order derivative are determined from initial condition. Then, with a small time increment Δt, the displacement function and its first-order derivative at the new position can be obtained using Eq. (27). This process continues to the end of the time limit.

5 Application

In this step application of IPM and HA is going to be indicated as follows:

5.1 Applying IPM

After introducing the variable $y=\frac{dQ}{dt}$, and Substituting $Q = A\cos(\omega t)$ into the Eq. (5), it is obtained:

$$\dot{y} = -\alpha A\cos(\omega t) - \beta A\cos^3(\omega t) \tag{29}$$

By using Fourier series expansion:

$$\dot{y} = \sum_{n=0}^{\infty}\alpha_{2n+1}\cos\left[(2n+1)\omega t\right] = \left[-\alpha A - \left(\frac{4}{\pi}\beta A^3\int_0^{\frac{\pi}{2}}\left(\cos^4(\phi)\right)d\phi\right)\right]\cos(\omega t) + \ldots \tag{30}$$

By integrating Eq. (30), and comparing with Eq. (12):

$$\omega_{IPM} = \sqrt{\alpha + \frac{3A^2\beta}{4}} \tag{31}$$

According to Eq. (31), the following approximate solution is obtained:

$$Q(t) = A\cos\left(\sqrt{\alpha + \frac{3A^2\beta}{4}}\ t\right) \tag{32}$$

5.2 Applying HA

Consider Eq. (5). Its Hamiltonian can be easily obtained, which reads:

$$H = \frac{1}{2}\dot{Q}^2 + \frac{1}{2}\alpha\, Q^2 + \frac{1}{4}\beta\, Q^4 \tag{33}$$

Integrating Eq. (33) with respect to t from 0 to $\frac{T}{4}$:

$$\bar{H} = \int_0^{\frac{T}{4}}\left\{\frac{1}{2}\dot{Q}^2 + \frac{1}{2}\alpha\, Q^2 + \frac{1}{4}\beta\, Q^4\right\}dt \tag{34}$$

With assumption that the solution can be expressed as:

$$Q\left(t\right)=A\cos\left(\omega t\right). \tag{35}$$

Substitute it to Eq. (34), leads to:

$$\begin{aligned}\bar{\mathrm{H}} &= \int_0^{\frac{T}{4}}\left(\frac{1}{2}A^2\omega^2\sin^2\left(\omega\, t\right)+\frac{1}{2}\alpha A^2\cos^2\left(\omega t\right)+\frac{1}{4}\beta A^4\cos^4\left(\omega t\right)\right)\\ &=\int_0^{\frac{\pi}{4}}\left(\frac{1}{2}A^2\,\omega\sin^2\left(\omega\,\mathrm{t}\right)+\frac{1}{2\,\omega}\alpha\,\mathrm{A}^2\cos^2\left(\omega\,\mathrm{t}\right)+\frac{1}{4\,\omega}\beta\,\mathrm{A}^4\cos^4\left(\omega\,\mathrm{t}\right)\right)\\ &=\frac{\pi}{8}\mathrm{A}^2\omega+\frac{\alpha\,\pi}{8\,\omega}A^2+\frac{\beta\,\pi}{64\,\omega}A^4\end{aligned} \tag{36}$$

Setting

$$\frac{\partial}{\partial A}\left(\frac{\partial\,\bar{H}}{\partial\left(1/\omega\right)}\right)=\frac{\pi}{4}A\;\omega^2+\frac{\alpha\pi}{4}A+\frac{4\beta\pi}{16}A^3=0 \tag{37}$$

From Eq. (37), it is obtained:

$$\omega_{HA}=\sqrt{\alpha+\frac{3\mathrm{A}^2\beta}{4}} \tag{38}$$

According to Eq. (38), the following approximate solution is resulted:

$$\mathrm{Q}\left(t\right)=A\cos\left(\sqrt{\alpha+\frac{3\mathrm{A}^2\beta}{4}}\;\mathrm{t}\right) \tag{39}$$

6 Results and Discussion

To illustrate and verify the accuracy of these new approximate methods, for the problem, a comparison of the time history oscillatory displacement responses with the numerical solution using Runge-Kutta method is presented in Table 1. Figure 2 represents a comparison of analytical solution of $Q(t)$ based on time between Iteration Perturbation Method (IPM) and Hamiltonian Approach (HA) for $\alpha=\pi/3, A=\pi/6$, Figure 3 shows comparison of analytical solution of dQ/dt based on time between Iteration Perturbation Method (IPM) and Hamiltonian Approach (HA) for $\alpha=\pi/3, A=\pi/6$, and Figures 4 and 5 are the comparisons of frequency corresponding to various parameters of α and A .

The effect of different parameters α and A are studied in Figures 6 and 7 simultaneously. It is evident that IPM and HA show an excellent agreement with the numerical solution and quickly convergent, and are valid for a wide range of vibration amplitudes and initial conditions. The accuracy of the results shows that the IPM and HA can be potentiality used for the analysis of strongly nonlinear oscillation problems accurately.

Table 1 Comparison between IPM and HA with Runge-Kutta method to various parameters of A , for t=3 (s), $\alpha=\pi$, $\beta=0.15$.

A	$Q_{IPM}=Q_{HA}$	$Q_{Runge\text{-}Kutta}$	Error percentage
π/18	0.09968	0.099667	0.013
π/9	0.201841	0.201743	0.048
π/6	0.308903	0.308571	0.107
π/3	0.681306	0.6787	0.383
π/2	1.164336	1.156396	0.686
π	3.126786	3.122405	0.140

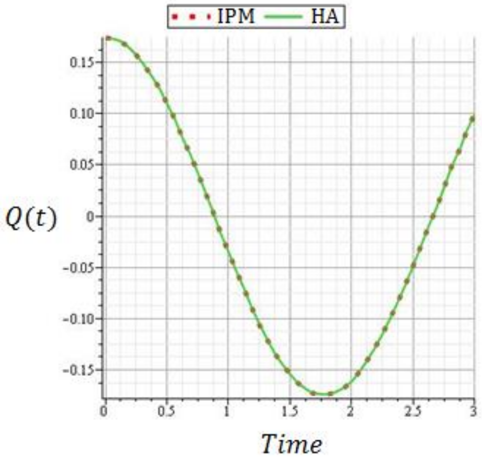

Figure 2 Comparison of analytical solution of $\mathbf{Q(t)}$ based on time with the numerical solution for $\alpha=\pi$, $\beta=0.15$, $A=\pi/18$.

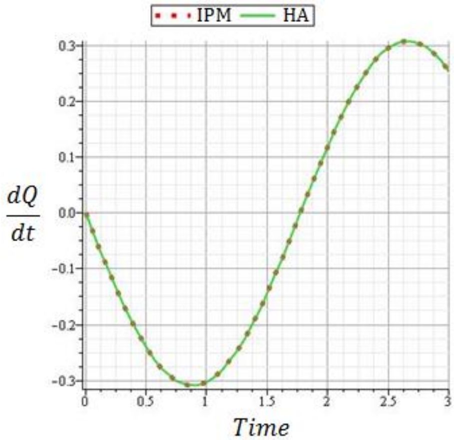

Figure 3 Comparison of analytical solution of $\mathbf{dQ/dt}$ based on time with the numerical solution for $\alpha=\pi$, $\beta=0.15$, $A=\pi/18$.

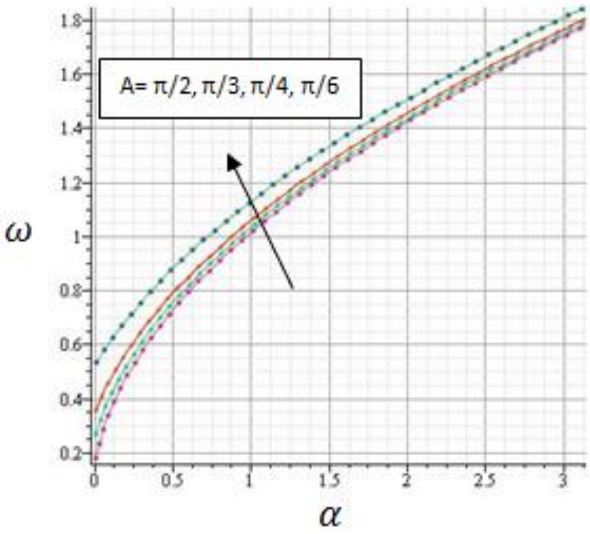

Figure 4 comparison of frequency corresponding to various parameters of (α) for β=0.15, A=π/2, π/3, π/4, π/6.
Solid Line: IPM,
Solid Box: HA.

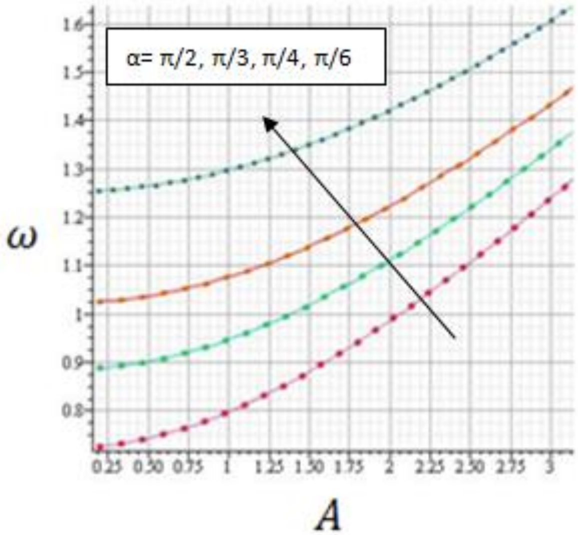

Figure 5 Comparison of frequency corresponding to various parameters of amplitude (A) for β=0.15, α=π/2, π/3, π/4, π/6.
Solid Line: IPM,
Solid Box: HA.

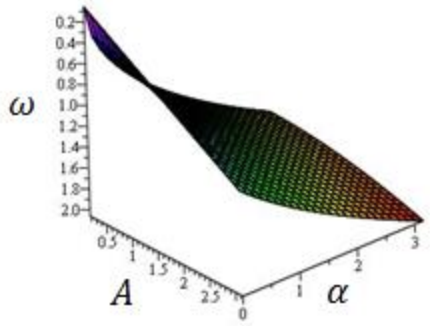

Figure 6 Sensitivity analysis of frequency for β=0.15 in IPM.

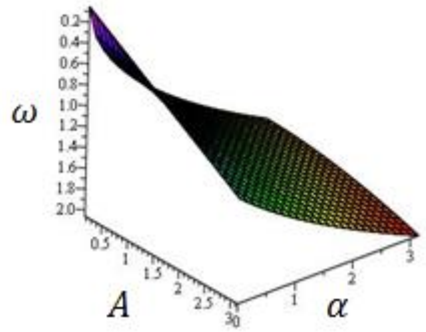

Figure 7 Sensitivity analysis of frequency for β=0.15 in HA.

7 Conclusion

In this study, two new methods called Iteration Perturbation Method (IPM) and Hamiltonian Approach (HA) have been used for non-natural oscillators. IPM and HA were utilized for analyzing the equation of the motion governing the nonlinear vibration of Euler-Bernoulli beams. It has been proved that the IPM and HA are very efficient, comfortable and sufficiently exact in engineering problems. IPM and HA can be simply extended to nonlinear equation for the analysis of nonlinear systems. The obtained results from the approximate analytical solutions are in excellent agreement with the corresponding exact solutions.

References

Bayat, M., Pakar, I., Bayat, M., (2011a). Analytical study on the vibration frequencies of tapered beams, Latin American Journal of Solids and Structures 8(2):149-162.

Bayat, M., Shahidi, M., Bayat, M., (2011b). Application of iteration perturbation method for nonlinear oscillators with discontinuities, Int. J. Phy. Sci 6(15):3608–3612.

He, J. H. (2012). Asymptotic Methods for Solitary Solutions and Compactons, Abstract and Applied Analysis 916793.

He, J. H. (2004). The homotopy perturbation method for nonlinear oscillators with discontinuities, Applied Mathematics and Computation 151:287–292.

Bayat, M., Pakar, I., Domairry, G., (2012). Recent developments of some asymptotic methods and their applications for nonlinear vibration equations in engineering problems: A review, Latin American Journal of Solids and Structures 9(2):145–234.

Ganji, D. D. (2006). The application of He's homotopy perturbation method to nonlinear equations arising in heat transfer, Phys. Lett. A. 355(4-5):337-341.

Beléndez, A., Pascual, C., Gallego, S., Ortuno, M., Neipp, C., (2007). Application of a modified He's homotopy perturbation method to obtain higher-order approximations of an $x^{1/3}$ force nonlinear oscillator, Physics Letters A 371(5-6):421-426.

Barari, A., Omidvar, M., Ghotbi, A. R., Ganji, D. D., (2008). Application of homotopy perturbation method and variational iteration method to nonlinear oscillator differential equations, Acta Applicandae Mathematicae 104(2):161–171.

Liao, S. J. (2003). Beyond perturbation: introduction to homotopy analysis method. Boca Raton (FL): Chapman & Hall/CRC Press.

Ghotbi, A. R., Bararnia, H., Domairry G., Barari, A., (2009). Investigation of a powerful analytical method into natural convection boundary layer flow, Commun Nonlinear Sci Numer Simulat 14(5):2222–2228.

Sohouli, A. R., Famouri, M., Kimiaeifar, A., Domairry, G., (2010). Application of homotopy analysis method for natural convection of Darcian fluid about a vertical full cone embedded in pours media prescribed surface heat flux, Commun Nonlinear Sci Numer Simulat 15(7):1691-1699.

Kimiaeifar, A., A. Saidi. R., Bagheri, G. H., Rahimpour, M., Domairry, D. G., (2009). Analytical solution for Van der Pol–Duffing oscillators, Chaos, Solitons and Fractals 42(5):2660–266.

He, J. H., Shou, D. H,. (2007). Application of parameter-expanding method to strongly nonlinear oscillators. International Journal of Nonlinear Sciences and Numerical Simulation 8:121–124. Xu, L. (2007). He's parameter-expanding methods for strongly nonlinear oscillators, J. Comput. Appl. Math. 207(1):148–154.

Kimiaeifar, A., Saidi, A. R., Sohouli, A. R., Ganji, D. D., (2010a). Analysis of modified Van der Pol's oscillator using He's parameter-expanding methods, Current Applied Physics 10(1):279–283.

Kimiaeifar, A., Lund, E., Thomsen, O. T., Barari, A. (2010b). On approximate analytical solutions of nonlinear vibrations of inextensible beams using parameter-expansion method, International Journal of Nonlinear Sciences and Numerical Simulation 11(9):743-753.

He, J. H. (1997). A new approach to nonlinear partial differential equations, Commun. Nonlinear., Sci., Numer. Simul. 2:230-235.

He, J. H. (2000). Variational iteration method for autonomous ordinary differential systems, Applied Mathematics and Computation 114:115-123.

Ganji, D. D., Nourollahi, M., Rostamin, M., (2007). A Comparison of Variational Iteration Method with Adomian's Decomposition Method in Some Highly Nonlinear Equations, International Journal of Science & Technology 2(2):179-188.

Barari, A., Kaliji, H. D, Ghadimi, M., Domairry, G., (2011). Non-linear vibration of Euler-Bernoulli beams, Latin American Journal of Solids and Structures 8(2):139-148.

Rafei, M., Ganji, D. D., Daniali, H., Pashaei, H., (2007). The variational iteration method for nonlinear oscillators with discontinuities, Journal of sound and vibration 305(4-5):614-620.

He, J. H. (2002). Preliminary report on the energy balance for nonlinear oscillations, Mechanics Research Communications, 29: 107–111.

Jamshidi, N., Ganji, D. D., (2010). Application of energy balance method and variational iteration method to an oscillation of a mass attached to a stretched elastic wire, Current Applied Physics 10(2):484–486.

Ganji, D. D., Gorji, M., Soleimani, S., Esmaeilpour, M., (2009). Solution of nonlinear cubic-quintic duffing oscillators uing he's energy balance method, Journal of Zhejiang University-Science A 10(9):1263–1268.

Fu, Y., Zhang, J., Wan, L., (2011). Application of the energy balance method to a nonlinear oscillator arising in the microelectromechanical system (mems), Current Applied Physics 11(3):482–485.

Afrouzi, G., Ganji, D. D., Talarposhti, R., (2011). He's energy balance method for nonlinear oscillators with discontinuities, International Journal of Nonlinear Sciences and Numerical Simulation 10(3):301–304.

Sfahani, M. G., Barari, A., Omidvar, M., Ganji, S. S., Domairry, G., (2011). Dynamic response of inextensible beams by improved energy balance method, Proceedings of the Institution of Mechanical Engineers, Part K Journal of multi-body Dynamics 225(1):66-73.

Momeni, M., Jamshidi, N., Barari, A., Ganji, D. D., (2011). Application of He's energy balance method to duffing harmonic oscillators, International Journal of Computer Mathematics 88(1):135–144.

Zhang, H. L., Xu, Y. G., Chang, J. R., (2009a). Application of He's energy balance method to a nonlinear oscillator with discontinuity, International Journal of Nonlinear Sciences and Numerical Simulation 10(2):207–214.

He, J. H. (2008). An improved amplitude-frequency formulation for nonlinear oscillators, International Journal of Nonlinear Sciences and Numerical Simulation 9(2):211.

Ganji, S. S., Ganji, D. D., Babazadeh, H., Sadoughi, N., (2010). Application of amplitude-frequency formulation to nonlinear oscillation system of the motion of a rigid rod rocking back, Mathematical Methods in the Applied Sciences 33(2):157–166.

Ren, Z. F., Liu, G. Q., Kang, Y. X., Fan, H. Y., Li, H. M., Ren. X. D., Gui, W. K., (2009). Application of he's amplitude Application of he's amplitude frequency formulation to nonlinerar oscillators with discontinuities, Physica Scripta 80:45003.

Zhang, H. L. (2009b). Application of he's amplitude-frequency formulation to a nonlinear oscillator with discontinuity, Computers & Mathematics with Applications 58(11-12):2197–2198.

He, J. H. (2001). Iteration perturbation method for strongly nonlinear oscillations, J. Vib. Control 7(5):631-642.

Ozis, T., Yildirim, A., (2009). Generating the periodic solutions for forcing van der pol oscillators by the iteration perturbation method, Nonlinear Analysis: Real World Applications 10(4):1984–1989.

Bayat, M., Pakar, I., (2011c). Nonlinear free vibration analysis of tapered beams by hamiltonian approach, Journal of vibroengineering 13(4):654–661.

Khan, Y., Wu, Q., Askari, H., Saadatnia, Z., Yazdi, M. K., (2010). Nonlinear vibration analysis of a rigid rod on a circular surface via hamiltonian approach, Mathematical and Computational Applications 15(5):974–977.

Xu, L. (2010). Application of hamiltonian approach to an oscillation of a mass attached to a stretched elastic wire, Mathematical and Computational Applications 15(5):901–906.

Yazdi, M. K., Khan, Y., Madani, M., Askari, H., Saadatnia, Z., Yildirim, A., (2010). Analytical solutions for autonomous conservative nonlinear oscillator, International Journal of Nonlinear Sciences and Numerical Simulation 11(11):975–980.

Yildirim, A., Saadatnia, Z., Askari, H., (2011a). Application of the hamiltonian approach to nonlinear oscillators with rational and irrational elastic terms, Mathematical and Computer Modelling 54(1-2):697-703.

Yildirim, A., Saadatnia, Z., Askari, H., Khan, Y., Yazdi, M. K., (2011b). Higher order approximate periodic solutions for nonlinear oscillators with the hamiltonian approach, Applied Mathematics Letters 24(12):2042-2051.

Wu, H. X., He, J. H., (2007). Solitary solutions periodic solutions and compacton-like solutions using EXP-function method, Comput. Math. Appl., 54(7–8):966–986

Davodi, A. G., Ganji, D. D., Davodi, A. G., Asgari A., (2010). Finding general and explicit solutions (2+1) dimensional Broer-Kaup-Kupershmidt system nonlinear equation by exp-function method, Applied Mathematics and computation 217(4):1415-1420.

Safari, M., Ganji, D. D., Moslemi, M., (2009). Application of He's variational iteration method and Adomian's decomposition method to the fractional KdV–Burgers–Kuramoto equation, Computers & Mathematics with Applications 58(11–12):2091-2097.

Sadighi, A., Ganji, D. D., (2007). Exact solutions of Laplace equation by homotopy-perturbation and Adomian decomposition methods, Physics Letters A 367(1–2):83-87.

Biazar, J., Porshokuhi, M. G., Ghanbari, B., (2010). Extracting a general iterative method from an Adomian decomposition method and comparing it to the variational iteration method, Computers & Mathematics with Applications 59(2):622-628.

Ghafoori, S., Motevalli, M., Nejad, M. G., Shakeri, F., Ganji, D. D., Jalaal, M., (2011). Efficiency of differential transformation method for nonlinear oscillation: Comparison with HPM and VIM, Current Applied Physics 11(4):965-971.

Joneidi, A. A., Ganji, D. D., Babaelahi, M., (2009). Differential Transformation Method to determine fin efficiency of convective straight fins with temperature dependent thermal conductivity, International Communications in Heat and Mass Transfer 36(7):757-762.

Borhanifar, A., Abazari, R., (2010). Numerical study of nonlinear Schrödinger and coupled Schrödinger equations by differential transformation method, Optics Communications 283(10):2026-2031.

He, J. H., (2007). Variational approach for nonlinear oscillators, Chaos, Solitons and Fractals 34(5):1430-1439.

Amani, A., Ganji, D. D., Jebelli, A. A., Shahabi, M., Nosar, N. S., (2011). Application of he's variational approach method for periodic solution of strongly nonlinear oscillation problems, International Journal of Applied Mathematics and Computation 2(3):33–43.

Ganji, S. S., Ganji, D. D., Babazadeh, H., Karimpour, S., (2008). Variational approach method for nonlinear oscillations of the motion of a rigid rod rocking back and cubic, Progress In Electromagnetics Research 4:23–32.

Shou, D. H. (2009). Variational approach for nonlinear oscillators with discontinuities, Computers & Mathematics with Applications 58(11-12):2416–2419.

Zhang, J. (2007). Variational approach to solitary wave solution of the generalized zakharov equation, Computers & Mathematics with Applications 54(7-8):1043–1046.

Beléndez, A., Gimeno, E., Alvarez M. L., Méndez, D. I., (2009). Nonlinear oscillator with discontinuity by generalized harmonic balance method, Computers & Mathematics with Applications 58(11–12):2117-2123.

He, J. H. (1999). Some new approaches to Duffing equation with strongly and high order nonlinearity (II) parametrized perturbation technique, Communications in Nonlinear Science and Numerical Simulation 4(1): 81-83.

He, J. H. (2006). Some asymptotic methods for strongly nonlinear equations, International Journal of Modern Physics B 20(10): 1141-1199.

Ding, X., Zhang, L., (2009). Applying he's parameterized perturbation method for solving differential-difference equation, International Journal of Nonlinear Sciences and Numerical Simulation 10(9):1249–1252.

Jalaal, M., Ghasemi, E., Ganji, D. D., Bararnia, H., Soleimani, S., Nejad, G. M., Esmaeilpour, M., (2011). Effect of temperature-dependency of surface emissivity on heat transfer using the parameterized perturbation method, Thermal Science, 15(1):123–125.

Lacarbonara, W. (1997). A Theoretical and Experimental Investigation of Nonlinear Vibrations of Buckled Beams, Master of Science Thesis, Virginia Politechnic Institute and State University.

Lestari, W., Hanagud, S., (2001). Nonlinear vibration of buckled beams: some exact solutions, International Journal of Solids and Structures 38(26):4741-4757.

Tse, F. S., Morse, I. E., Hinkle, R. T., (1978). Mechanical Vibrations: Theory and Applications, Second ed. Allyn and Bacon Inc.,boston.

He, J. H., (2002). Preliminary report on the energy balance for nonlinear oscillations, Mechanics Research Communications 29:107–111.

He, J. H., (2010). Hamiltonian approach to nonlinear oscillators, Physics Letters A 374:2312–2314.