Latin American Journal of
Solids and Structures
www.lajss.org

A General Symplectic Method for the Response Analysis of Infinitely Periodic Structures Subjected to Random Excitations

Abstract

A general symplectic method for the random response analysis of infinitely periodic structures subjected to stationary/non-stationary random excitations is developed using symplectic mathematics in conjunction with variable separation and the pseudo-excitation method (PEM). Starting from the equation of motion for a single loaded substructure, symplectic analysis is firstly used to eliminate the dependent degrees of the freedom through condensation. A Fourier expansion of the condensed equation of motion is then applied to separate the variables of time and wave number, thus enabling the necessary recurrence scheme to be developed. The random response is finally determined by implementing PEM. The proposed method is justified by comparison with results available in the literature and is then applied to a more complicated time-dependent coupled system.

Keywords

Infinitely periodic structure; Symplectic mathematics; Variable separation; Pseudo-excitation method; Random vibration

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Received 22 Mar 2012;
In revised form 08 May 2012

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1 INTRODUCTION

Infinitely periodic structures are widely used in engineering practice, e.g. railway tracks, multi-span bridges and petroleum pipe-lines. They consist of identical substructures that are joined together to form a continuous structure. In recent decades, much attention has been paid to such structures and many important advances have been made, mainly in the areas of vibration characteristics, free vibration propagation and forced vibration induced by stationary harmonic loads [4, 11–16, 18, 19, 21, 22, 24, 25]. In particular, symplectic mathematics has been applied successfully [21, 22, 24, 25] to provide a precise and efficient approach for investigating the dynamic response and wave propagation caused by harmonic forces. Subsequently Lin *et al.* derived the stationary/non-stationary random response by means of the pseudo-excitation method (PEM) [5–8] and Lu *et al.* [9] applied this work to the random vibration analysis of coupled vehicle-track systems with the fixed-vehicle model, which considerably reduced the number of degrees of freedom (DOFs) required to describe the track.

However, vibration of infinitely periodic structures subjected to arbitrary excitation has received much less attention. Belotserkovskiy [1] investigated an infinitely periodic beam subjected to a moving harmonic load by analyzing one beam segment between neighboring supports with boundary conditions derived from Bernoulli-Euler beam theory and this was later extended to deal with infinitely periodic strings [2, 3] ; Sheng *et al.* [17] proposed a wave number-based approach to study a two-and-a-half-dimensional finite-element model subjected to a moving or stationary harmonic load, while Mead's [10] latest advance presents a general theory for the forced vibration of multi-coupled, one-dimensional periodic structures by firstly analyzing the semi-infinite periodic system excited only at its end, which is then connected to either side of the loaded substructure. The present authors [20], based on the work of Lu *et al.* [9], selected a series of wave numbers evenly distributed in the interval $[0, 2\pi)$ and derived the corresponding propagation constants. This enabled the random response of the infinitely periodic structures to be obtained by accumulating the pass-band frequency responses. Such an approach, when combined with PEM, results in an efficient method for computing response PSDs of vehicle-track coupled systems based on the moving-vehicle model. However, one drawback stems from the discreteness of wave numbers, which inevitably causes discrete numerical errors.

In order to eliminate this problem and substantially improve the technique, a continuous integration is used as follows in this paper yield to a new and general approach for the response analysis of infinitely periodic structures subjected to arbitrary excitations. This new method is based on a symplectic mathematical scheme combined with a variable separation approach in which only the loaded substructure is included in the calculation. The dependent DOFs are firstly condensed into the independent ones according to the properties of the wave propagation constants. The condensed equation of motion is then derived, in which the coefficient matrices are functions of the wave number. By applying Fourier expansions to these coefficient matrices and the response vectors, the time and wave number variables are easily separated and a recurrence scheme is developed accordingly. Finally, in accordance with the work of Lin *et al.*[5, 6], the resulting equations are combined with PEM for stationary or non-stationary random response analysis, after which the response power spectral densities (PSDs) and the standard deviations can be derived conveniently. The proposed method is justified by comparison with a numerical example in Reference [6] and the theory is then applied to the random analysis of a mass moving on a rail that is supported periodically ad infinitum.

2 SYMPLECTIC ANALYSIS FOR AN INFINITELY PERIODIC STRUCTURE SUBJECTED TO ARBITRARY LOADS

In this section, the symplectic mathematical scheme is generalized to investigate the response of an infinitely periodic structure subjected to arbitrary loads. The infinitely periodic structure shown in Figure 1 consists of two kinds of substructures, denoted as sub and sub*, which are identical except that sub* is subjected to an arbitrary load $f(t)$.

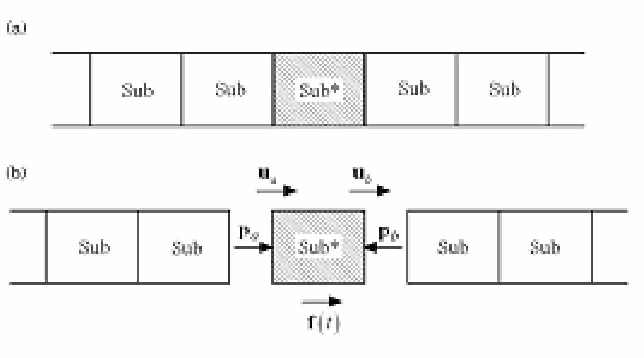

Figure 1 Infinitely periodic structure showing the loaded substructure sub* and the forces and displacements at its interfaces with its neighbours.

The equation of motion for this substructure is

$$M\ddot{u} + C\dot{u} + Ku = f(t) + f_b \tag{1}$$

in which: **M**, **C** and **K** are the $n \times n$ mass, damping and stiffness matrices that can be created by any means;

$$u = \left\{ \begin{array}{ccc} u_a^{\mathrm{T}} & u_b^{\mathrm{T}} & u_i^{\mathrm{T}} \end{array} \right\}^{\mathrm{T}}; f_b = \left\{ \begin{array}{ccc} p_a^{\mathrm{T}} & -p_b^{\mathrm{T}} & 0 \end{array} \right\}^{\mathrm{T}}; \tag{2}$$

where: superscript T denotes transpose; u_a and u_b are the displacement vectors at the left- and right-hand interface, see Fig 1; u_i is the internal displacement vector and; p_a and p_b are the corresponding nodal force vectors on the interfaces.

For an undamped and unloaded substructure, it has been proven in References [21, 22, 24, 25] that

$$\left\{ \begin{array}{c} u_b \\ p_b \end{array} \right\} = S \left\{ \begin{array}{c} u_a \\ p_a \end{array} \right\} = \mu \left\{ \begin{array}{c} u_a \\ p_a \end{array} \right\} \tag{3}$$

in which S is a frequency-dependent symplectic transfer matrix that has eigenvalues μ and satisfies the symplectic orthogonality relationships

$$S^{\mathrm{T}} J_n S = J_n; \quad J_n = \left[\begin{array}{cc} 0 & I_n \\ -I_n & 0 \end{array} \right]; \quad J_n^{\mathrm{T}} = J_n^{-1} = -J_n \tag{4}$$

where: I_n is the n-dimensional unit matrix and; the μ are known as the wave propagation constants, where $|\mu| = 1$ refers to transmission waves that propagate without decay, i.e. they lie within the frequency pass-band. μ can be expressed as

$$\mu = e^{j\theta}; \quad j = \sqrt{-1} \tag{5}$$

in which θ is the wave number and lies in the interval $[0, 2\pi)$.

Let

$$T(\theta) = T = \left[\begin{array}{cc} I_n & 0 \\ e^{j\theta} I_n & 0 \\ 0 & I_n \end{array} \right] \tag{6}$$

Then for each wave number θ, it can be verified that

$$\left\{\begin{array}{c} u_a^* \\ u_b^* \\ u_i^* \end{array}\right\} = T\left\{\begin{array}{c} u_a^* \\ u_i^* \end{array}\right\}; T^{\mathrm{H}}\left\{\begin{array}{c} p_a^* \\ -p_b^* \\ 0 \end{array}\right\} = \left\{\begin{array}{c} p_a^* - e^{-j\theta} p_b^* \\ 0 \end{array}\right\} = \left\{\begin{array}{c} 0 \\ 0 \end{array}\right\} \tag{7}$$

in which: superscript H denotes complex conjugate transpose; u_a^*, u_b^* and u_i^* are the response vectors related to a given wave number and p_a^* and p_b^* are the corresponding nodal force vectors and hence are functions of wave number θ and time t. Substituting Eq. (7) into Eq. (1) and pre-multiplying both sides by T^{H} gives the condensed equation of motion of the loaded substructure as

$$\bar{M}^*(\theta)\,\ddot{\bar{u}}^*(\theta,t) + \bar{C}^*(\theta)\,\dot{\bar{u}}^*(\theta,t) + \bar{K}^*(\theta)\,\bar{u}^*(\theta,t) = T^{\mathrm{H}}(\theta)\,f(t) \tag{8}$$

in which

$$\bar{u}^* = \left\{\begin{array}{cc} u_a^{*\mathrm{T}} & u_i^{*\mathrm{T}} \end{array}\right\}^{\mathrm{T}}; \bar{M}^* = T^{\mathrm{H}}MT; \bar{C}^* = T^{\mathrm{H}}CT; \bar{K}^* = T^{\mathrm{H}}KT \tag{9}$$

Note that the natural frequencies of the infinitely periodic structure can be obtained by solving the following generalized eigenproblem [16]

$$\bar{K}^*\Psi = \bar{M}^*\Psi\Omega^2 \tag{10}$$

in which: Ω is the diagonal matrix of natural frequencies and Ψ is the corresponding modal matrix. The number of natural frequencies developed from each wave number is equal to the number of independent DOFs of the substructure. Since there are infinitely many wave numbers, an infinitely periodic structure yields an infinite number of natural frequencies. In Reference [20] a finite number of wave numbers, evenly distributed in the interval $[0, 2\pi)$, were selected to calculate the responses. This inevitably results in the discrete errors mentioned previously. However, this is circumvented below by performing a continuous integration instead.

Assume that the response of each substructure can be determined by performing the following integration.

$$u_k(t) = \frac{1}{2\pi}\int_0^{2\pi} T(\theta)\,\bar{u}^*(\theta,t)\,e^{jk\theta}d\theta; \quad (k = 0, \pm1, \pm2\cdots) \tag{11}$$

where $k = 0$, $k > 0$, $k < 0$ correspond, respectively, to the loaded substructure and the substructures to its right and left. However, $\bar{u}^*(\theta,t)$ cannot be solved from Eq. (8) directly and so the following approach is used instead.

Let the matrices $\bar{M}^*$, $\bar{C}^*$, $\bar{K}^*$ and T be expressed as

$$\begin{array}{l} \bar{M}^* = \bar{M}_0 + \bar{M}_1 e^{j\theta} + \bar{M}_{-1}e^{-j\theta}; \;\; \bar{C}^* = \bar{C}_0 + \bar{C}_1 e^{j\theta} + \bar{C}_{-1}e^{-j\theta} \\ \bar{K}^* = \bar{K}_0 + \bar{K}_1 e^{j\theta} + \bar{K}_{-1}e^{-j\theta}; \;\; T = T_0 + T_{-1}e^{j\theta} \end{array} \tag{12}$$

in which

$$\bar{M}_0 = \begin{bmatrix} M_{aa} + M_{bb} & M_{ai} \\ M_{ia} & M_{ii} \end{bmatrix}; \quad \bar{M}_1 = \begin{bmatrix} M_{ab} & 0 \\ M_{ib} & 0 \end{bmatrix}; \quad \bar{M}_{-1} = \begin{bmatrix} M_{ba} & M_{bi} \\ 0 & 0 \end{bmatrix}$$
$$T_0 = \begin{bmatrix} I_n & 0 \\ 0 & 0 \\ 0 & I_n \end{bmatrix}; \quad T_{-1} = \begin{bmatrix} 0 & 0 \\ I_n & 0 \\ 0 & 0 \end{bmatrix} \tag{13}$$

where: $M_{lm}\,(l, m = a, b, i)$ are the submatrices corresponding, respectively, to the DOFs at the two interfaces and the internal DOFs and; the submatrices of $\bar{C}$ and $\bar{K}$ are defined analogously to those of $\bar{M}$. Now $\bar{u}^*(\theta, t)$ can be expressed as the sum of an infinite number of spatial harmonics by using Fourier expansion to give

$$\bar{u}^* = \sum_n \bar{u}_{en} e^{jn\theta}; \quad (n = 0, \pm 1, \pm 2 \cdots) \tag{14}$$

in which $\bar{u}_{en}$ $(n = 0, \pm 1, \pm 2 \cdots)$ denotes the Fourier expansion coefficients. Eq. (11) can then be rewritten as

$$u_k(t) = T_0 \bar{u}_{e(-k)} + T_{-1} \bar{u}_{e(-k-1)}; \quad (k = 0, \pm 1, \pm 2 \cdots) \tag{15}$$

Substituting Eqs. (12) and (14) into Eq. (8) and separating the variables of time and wave number by using the orthogonality of the exponents gives

$$\begin{bmatrix} \hat{M}_{mm} & \hat{M}_{ms} \\ \hat{M}_{sm} & \hat{M}_{ss} \end{bmatrix} \begin{Bmatrix} \ddot{\hat{u}}_{mk} \\ \ddot{\hat{u}}_s \end{Bmatrix} + \begin{bmatrix} \hat{C}_{mm} & \hat{C}_{ms} \\ \hat{C}_{sm} & \hat{C}_{ss} \end{bmatrix} \begin{Bmatrix} \dot{\hat{u}}_{mk} \\ \dot{\hat{u}}_s \end{Bmatrix} + \begin{bmatrix} \hat{K}_{mm} & \hat{K}_{ms} \\ \hat{K}_{sm} & \hat{K}_{ss} \end{bmatrix} \begin{Bmatrix} \hat{u}_{mk} \\ \hat{u}_s \end{Bmatrix} = \begin{bmatrix} F_{mk} \\ 0 \end{bmatrix} f(t) \quad (k = 1, 2, \cdots) \tag{16}$$

in which

$$\hat{u}_{mk} = \left\{ \bar{u}_{e0}^{\mathrm{T}} \;\; \bar{u}_{e1}^{\mathrm{T}} \;\; \bar{u}_{e-1}^{\mathrm{T}} \;\; \cdots \;\; \bar{u}_{ek}^{\mathrm{T}} \;\; \bar{u}_{e-k}^{\mathrm{T}} \right\}^{\mathrm{T}}; \quad \hat{u}_s = \left\{ \bar{u}_{e(k+1)}^{\mathrm{T}} \;\; \bar{u}_{e-(k+1)}^{\mathrm{T}} \;\; \cdots \right\}^{\mathrm{T}};$$
$$\hat{M}_{mm} = \begin{bmatrix} \bar{M}_0 & \bar{M}_{-1} & \bar{M}_1 & & & \\ \bar{M}_1 & \bar{M}_0 & & \bar{M}_{-1} & & \\ \bar{M}_{-1} & & \bar{M}_0 & & \ddots & \\ & \bar{M}_1 & & \ddots & & \bar{M}_1 \\ & & \ddots & & & \\ & & & \bar{M}_{-1} & & \bar{M}_0 \end{bmatrix}; \quad F_{mk} = \begin{bmatrix} T_0^{\mathrm{T}} \\ 0 \\ T_{-1}^{\mathrm{T}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{17}$$
$$\hat{M}_{ss} = \begin{bmatrix} \bar{M}_0 & & \bar{M}_{-1} & & \\ & \bar{M}_0 & & \bar{M}_1 & \\ \bar{M}_1 & & \bar{M}_0 & & \ddots \\ & \bar{M}_{-1} & & \bar{M}_0 & \\ & & \ddots & & \ddots \end{bmatrix}; \quad \hat{M}_{ms} = \begin{bmatrix} 0 & & & \cdots & 0 \\ \vdots & & & & \\ 0 & & & \iddots & \vdots \\ \bar{M}_{-1} & 0 & & & \\ & \bar{M}_1 & 0 & \cdots & 0 \end{bmatrix} = M_{sm}^{\mathrm{T}}$$

and $\hat{C}$ and $\hat{K}$ can be substituted for $\hat{M}$ throughout Eq. (17).

By inspection it can be seen that: $\hat{u}_{mk}$ is a finite-dimensional vector; $\hat{u}_s$ is of infinite-dimension and; the matrices of Eq. (16) are very sparse. Thus for each value of k, Eq. (16) can be rewritten in block form as

$$\hat{M}_{mm}\ddot{\hat{u}}_{mk} + \hat{C}_{mm}\dot{\hat{u}}_{mk} + \hat{K}_{mm}\hat{u}_{mk} = F_{mk}f\left(t\right) - M_{ms}\ddot{u}_s - C_{ms}\dot{u}_s - K_{ms}u_s \tag{18}$$

$$M_{ss}\ddot{\hat{u}}_{sk} + C_{ss}\dot{\hat{u}}_{sk} + K_{ss}\hat{u}_{sk} = -M_{sm}\ddot{u}_{mk} - C_{sm}\dot{u}_{mk} - K_{sm}u_{mk} \tag{19}$$

where: $u_s = \left\{ \begin{array}{cc} \bar{u}_{e(k+1)}^{\mathrm{T}} & \bar{u}_{e-(k+1)}^{\mathrm{T}} \end{array} \right\}^{\mathrm{T}}$; $u_{mk} = \left\{ \begin{array}{cc} \bar{u}_{ek}^{\mathrm{T}} & \bar{u}_{e-k}^{\mathrm{T}} \end{array} \right\}^{\mathrm{T}}$ and; M_{ms}, M_{sm}, C_{ms}, C_{sm}, K_{ms} and K_{sm} are submatrices. Noting that Eq. (19) is of infinite-dimension, it needs to be calculated in truncated form. Since its coefficient matrices remain unchanged irrespective of the value of k, Eq. (19) can be transformed into state-space as [23]

$$\dot{v}_s = H_s v_s + Q v_{mk} \tag{20}$$

in which $v_s = \left\{ \begin{array}{cc} \hat{u}_s^{\mathrm{T}} & \dot{\hat{u}}_s^{\mathrm{T}} \end{array} \right\}^{\mathrm{T}}$; $v_{mk} = \left\{ \begin{array}{cc} u_{mk}^{\mathrm{T}} & \dot{u}_{mk}^{\mathrm{T}} \end{array} \right\}^{\mathrm{T}}$; H_s is a Hamiltonian matrix and; Q is the load coefficient matrix. Usually, Eq. (20) is solved using a step-by-step integration scheme. Thus if the response at time t is known, the response at time $t + \Delta t$ can be expressed as

$$v_s\left(t + \Delta t\right) = T_s\left(\Delta t\right) v_s\left(t\right) + R v_{mk}\left(t\right) \tag{21}$$

where $T_s(\Delta t)$ is an exponential matrix whose precise computation is described in Reference [23] and the physical meaning of the $n-th$ column of matrix R is the response $v_s\left(t+\Delta t\right)$ when assuming that $v_s\left(t\right) = 0$ and that the $n-th$ value of $v_{mk}\left(t\right)$ is 1 while all others are zero. Consequently, the responses can be computed by the following recurrence scheme: (1) let $k = 1$ and solve Eqs. (18) and (21) by using step-by-step integration to obtain the responses $\bar{u}_{e0}$, $\bar{u}_{e1}$ and $\bar{u}_{e-1}$; (2) Similarly, let $k = 2$ and substitute $\bar{u}_{e0}$, $\bar{u}_{e1}$ and $\bar{u}_{e-1}$ into Eq. (18) to obtain $\bar{u}_{e2}$ and $\bar{u}_{e-2}$ and; (3) Compute the remaining responses similarly and hence find the responses of the substructures by using Eq. (15).

Note that the method is still applicable if the coefficient matrices of Eq. (1) are time-dependent, e.g. due to a moving mass coupling with the infinitely periodic structure.

3 RESPONSES OF INFINITELY PERIODIC STRUCTURES SUBJECTED TO RANDOM EXCITATIONS

PEM is an accurate and highly efficient algorithm for structural stationary or non-stationary random response analysis. In this section, it is combined with the above method to find the random responses. Consider the most complicated case of a time-dependent system excited by an evolutionary random point excitation. Then the equation of motion of the system is

$$\begin{gathered} M\ddot{u} + C\dot{u} + Ku = f\left(t\right) + f_b \\ f\left(t\right) = r(t) g\left(t\right) x\left(t\right) \end{gathered} \tag{22}$$

in which: M, C and K are functions of time; $r(t)$ identifies which element is being excited; $g\left(t\right)$ is the modulation function and; $x\left(t\right)$ is a stationary random process with PSD $S_{xx}\left(\omega\right)$.

The corresponding response vector can be expressed by the convolution integral

$$u(t) = \int_0^t H(t,\tau)\, f(\tau)\, d\tau \tag{23}$$

in which $H(t,\tau)$ is the frequency response matrix. Multiplying $u(t)$ by its transpose and applying the mathematical expectation operator, the variance matrix of the response vector is given by

$$\begin{aligned} R_{uu}(t) &= \mathrm{E}\left[u(t)u^{\mathrm{T}}(t)\right] = \textstyle\int_0^t\int_0^t H(t,\tau_1)\,\mathrm{E}\left[f(\tau_1)\,f^{\mathrm{T}}(\tau_2)\right] H^{\mathrm{T}}(t,\tau_2)\, d\tau_1 d\tau_2 \\ &= \textstyle\int_0^t\int_0^t H(t,\tau_1)\, r(\tau_1) r^{\mathrm{T}}(\tau_2)\, H^{\mathrm{T}}(t,\tau_2)\, g(\tau_1)\, g(\tau_2)\,\mathrm{E}\left[x(\tau_1)\, x(\tau_2)\right] d\tau_1 d\tau_2 \end{aligned} \tag{24}$$

According to the Wiener - Khintchine theorem

$$\mathrm{E}\left[x(\tau_1)\, x(\tau_2)\right] = R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{i\omega(\tau_2-\tau_1)} d\omega \tag{25}$$

Substituting Eq. (25) into Eq. (24) and exchanging the integral order gives the evolutionary PSD matrix of the response vector $u(t)$ as

$$R_{uu}(t) = \int_{-\infty}^{+\infty} S_{uu}(\omega,t) d\omega \tag{26}$$

where

$$S_{uu}(\omega,t) = \int_0^t \int_0^t H(t,\tau_1)\, r(\tau_1) r^{\mathrm{T}}(\tau_2)\, H^{\mathrm{T}}(t,\tau_2)\, g(\tau_1)\, g(\tau_2)\, S_{xx}(\omega)\, e^{i\omega(\tau_2-\tau_1)} d\tau_1 d\tau_2 \tag{27}$$

It can be seen that Eq. (27) is a double integral expression which is very time consuming to compute directly. Therefore, PEM is used instead. Assume that the structure is subjected to a pseudo-excitation

$$\tilde{f}(\omega,t) = r(t) g(t) \sqrt{S_{xx}(\omega)} \mathrm{e}^{i\omega t} \tag{28}$$

Eq. (26) can then be rewritten as

$$S_{uu}(\omega,t) = \tilde{u}^{*}(\omega,t)\tilde{u}^{\mathrm{T}}(\omega,t); \tilde{u}(\omega,t) = \int_0^t H(t,\tau)\, \tilde{f}(\omega,\tau)\, d\tau \tag{29}$$

where the superscript * denotes complex conjugate. It is clear that $\tilde{u}(\omega,t)$ is the response of the structure when it is subjected to the pseudo-excitation and also that the first of Eqs. (29) has a much simpler form than Eq. (27). Thus the use of PEM to transform random excitations into harmonic pseudo-excitations leads to a very significant reduction in computational effort.

Substituting the pseudo excitation of Eq. (28) into Eq. (18) enables the pseudo responses of the infinitely periodic structure to be obtained using the above recurrence scheme. Denoting the pseudo response of the response $u(t)$ as $\tilde{u}(\omega,t)$ and utilizing PEM, the PSD of $u(t)$ can be written as

$$S(\omega,t) = \tilde{u}(\omega,t)\ \tilde{u}^{*}(\omega,t) \tag{30}$$

It is clear that if M, C and $K$are time-independent, the system degenerate into a time-independent one, and if $g(t) = 1$, the random excitation degenerates into a stationary one. PEM is still applicable in these cases.

4 NUMERICAL EXAMPLES

4.1 Example 1: Correctness verification

In this section, the proposed method is justified by comparison with the method proposed in Reference [6].

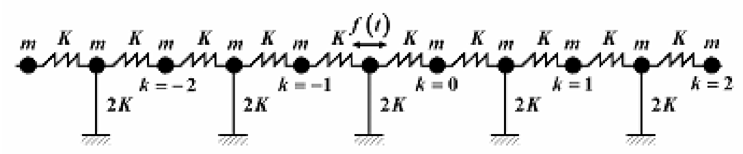

Figure 2 The infinitely periodic structure of Example 1 which is subjected to the point evolutionary random excitation $f(t)$. It consists of cantilever columns with stiffness $2K$ for lateral displacements at their upper ends and which carry masses m which are connected by two springs of stiffness $K$with a mass m where they are connected together.

Consider the infinitely periodic structure defined in Figure 2 and its caption, subjected to an evolutionary random excitation given by

$$f(t) = g(t)\,x(t) \tag{31}$$

in which the modulation function $g(t)$ has the form shown in Figure 3, i.e.

$$g(t) = \begin{cases} 0.1t & when\ \ 0 \le t \le 10 \\ 1.0 & when\ \ 10 < t \le 40 \\ 0.1(50-t) & when\ 40 < t \le 50 \\ 0 & otherwise \end{cases} \tag{32}$$

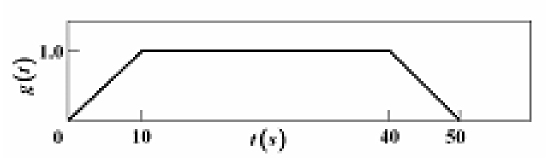

Figure 3 Envelope function $g(t)$

$x(t)$ is considered as a band-limited white noise, its units being $N^2 s$

$$S_{xx}(\omega) = \begin{cases} 1.0 & when & |\omega| \le \omega_0 \\ 0.0 & when & |\omega| > \omega_0 \end{cases} \tag{33}$$

The calculations used $K = 1$; $m = 1$; $\omega_0 = 3$ and the hysteretic damping factor $\nu = 0.1$.

Figure 4 gives the time dependent variances of the displacements at stations $k = 0$, 1 and 2, with the results from the proposed method shown as the solid line, while those from the theory of Reference [6] are given by the asterisks. Clearly the results agree very well and the difference of the peak values at point A is less than 0.01%, which justifies the correctness of the proposed method.

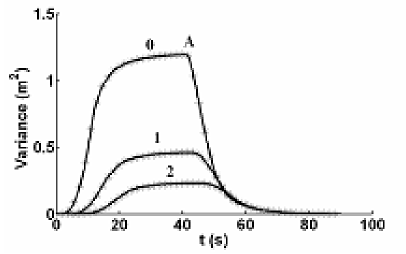

Figure 4 Time-dependent variances of the displacements at stations 0, 1 and 2 of Example 1.

4.2 Example 2: Application to a time-dependent coupled system

In this example, the proposed method is applied to find the time-dependent random responses when a mass of $1000kg$ crosses an infinite periodically supported rail/sleeper/ballast system at a velocity of $100km/h$, see Figure 5. The track irregularity is regarded as white noise with PSD $S_{rr}(\omega) = 1.0\left(m^2/rad/s\right)$ and the parameters of the system are listed in Table 1.

Table 1 Parameters, defined in Figure 5, of the periodically supported rail of Example 2.

Bending stiffness EI	$6.62 \times 10^6 \mathrm{N\,m}^2$	Stiffness K_b	$1.82 \times 10^8 \mathrm{N/m}$
Rail mass/length ρA	60.64kg/m	Stiffness K_f	$1.47 \times 10^8 \mathrm{N/m}$
Spacing l	0.545m	Damping C_p	$7.5 \times 10^4 \mathrm{N\,s/m}$
Mass M_s	237kg	Damping C_b	$5.88 \times 10^4 \mathrm{N\,s/m}$
Mass M_b	1478kg	Damping C_f	$3.115 \times 10^4 \mathrm{N\,s/m}$
Stiffness K_p	$1.2 \times 10^8 \mathrm{N/m}$		

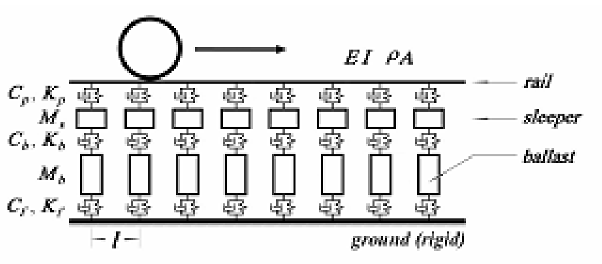

Figure 5 Example 2: A mass moving on a rail which is supported by the sleepers, ballast and spring and dashpot systems shown.

Figure 6 gives the PSD and variance of one static point at a support on the rail as the mass passes it. It can be seen that, as might be expected, the responses are largest at high load frequencies and when the moving mass is close to the point. The same conclusions are drawn when the static point was taken midway between supports and the results are not shown because they are very similar to Figure 6, e.g. the peak on Figure 6(b) was reduced by 10.66%. Such examples could be extended without difficulty to allow for train wheels attached to bogies moving on the track.

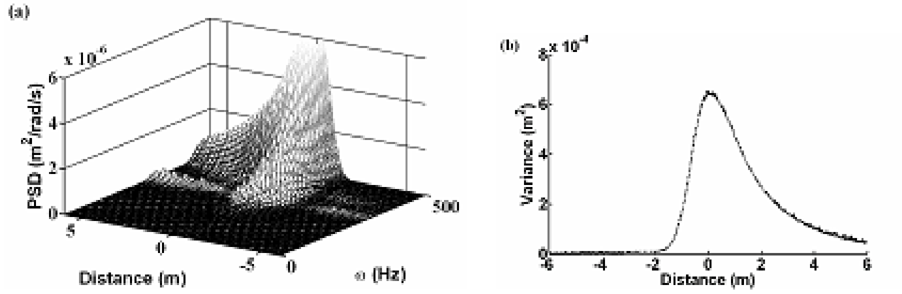

Figure 6 Vertical displacement responses of a static point at a support on the rail of Example 2. (a) PSD, (b) Variance.

5 CONCLUSIONS

Based on symplectic mathematics, a condensed equation of motion has been established for the loaded substructure of an infinitely periodic structure, the coefficient matrices of which are functions of the wave number. A Fourier expansion was then applied to separate the variables of time and wave number, which led to a recurrence scheme for computing the responses of the infinitely periodic structure. Finally, this method was combined with PEM to yield a convenient method for analyzing the random vibration of the structure. The proposed method was justified by a numerical example and was then applied to a more complicated time-dependent coupled system.

Acknowledgments Support for this work from the National Natural Science Foundation of China (Nos. 10972048, 90815023), the National 973 Plan Project (No. 2010CB832700), the State Science and Technology Supporting Program (No. 2009BAG12A04-A03) and the Cardiff Advanced Chinese Engineering Centre are gratefully acknowledged.

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