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Geometry and Topology Optimization of Statically Determinate Beams under Fixed and Most Unfavorably Distributed Load

Abstract

The paper concerns topology and geometry optimization of statically determinate beams with an arbitrary number of pin supports. The beams are simultaneously exposed to uniform dead load and arbitrarily distributed live load and optimized for the absolute maximum bending moment. First, all the beams with fixed topology are subjected to geometrical optimization by genetic algorithm. Strict mathematical formulas for calculation of optimal geometrical parameters are found for all topologies and any ratio of dead to live load. Then beams with the same minimal values of the objective function and different topologies are classified into groups called topological classes. The detailed characteristics of these classes are described.

Keywords

Statically determinate beams, geometry and topology optimization, genetic algorithm, stationary load, most unfavorable load.

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NOMENCLATURE

$\mathbf{b}_i^n$, b_i	assignment of lengths l_{Bj} to left or right side of supports, $i = 1, 2, ..., n$
c_E, c_H	number of external and internal cantilevers
c_{Bj}	number of j-th cantilevers from the top of the beam interaction scheme, $j = 1, 2, \ldots n-1$
$\mathbf{g}$	beam chromosome
$\mathbf{h}$	coordinates of hinges
l, l_E, l_H, L	lengths of optimal beam segments and length of beam, see Fig. 1
l_{Bj}	distance from intersection of optimal moment diagram (with maximum value M_i^n at the bottom) with beam axis to the nearest support, $j = 1, 2, \ldots n-1$, see Fig. 1 and Fig. 2
M_i, M_i^n	optimal moment value of topology $\mathbf{t}_i$ and class $\mathbf{T}_i^n$

NOMENCLATURE (continuation)

n	number of supports
p^n , $p^{2:n}$	number of topological classes in set $\mathbf{T}^n$ and $\mathbf{T}^{2:n}$
q	dimensionless intensity of uniformly distributed gravity dead load for constant sum of gravity dead load and maximum gravity live load intensities, equal to one, $0 \le q \le 1$
$1-q$	maximum dimensionless intensity of arbitrarily distributed live load for constant sum of both load intensities, equal to one
R	equivalence relation of beam topologies
$\mathbf{s}$	coordinates of supports
$\mathbf{t}_i$	beam topology
t_i	topological code of support i, $i=1,2,...,n$
$\mathbf{T}^n$, $\mathbf{T}^{2:n}$	set of all topologies with n supports and with two to n supports
$\mathbf{T}_i^n$, $\mathbf{T}_i^{2:n}$	topological class with n supports and with two to n supports
x	axial coordinate
y_i	dimensionless length of cantilever, $i=1,2,...,n$
z_i	dimensionless length of span, $i=1,2,...,n$
$(\cdot)^n$, $(\cdot)_i^n$	quantities in set $\mathbf{T}^n$ and class $\mathbf{T}_i^n$

1 INTRODUCTION

Structural optimization, which includes sizing, geometry, and topology optimization, has been a very common topic of research. Topology optimization is a relatively new but fast growing field of this research (Kirsch, 1989; Rozvany et al., 1995; Eschenauer and Olhoff, 2001; Fancello and Pereira, 2003; Marczak, 2008; Rozvany, 2009; Lopes et al., 2015). In recent years, structural topology optimization has received a boost due to the recognition that topological parameters can lead to a significant improvement in the quality of structures. Thanks to the widespread availability of high-speed computers and the development of powerful computational methods for the structural analysis, scientists can return to problems considered to be investigated and can make new interesting discoveries. The topological optimization of statically determinate beams is such an insufficiently explored problem.

Beams have been widely used in civil and mechanical engineering and their optimization has been extensively studied in the literature. Studies in beam optimization originated many years ago and are attributed to Galileo Galilei who dealt with an improvement in the shape of statically determinate beams (Timoshenko, 1953). Since then, many researchers were engaged in the optimization of statically determinate and indeterminate beams for different objective functions and loading conditions (Mróz and Rozvany, 1975; Imam and Al-Shihri, 1996; Wang and Chen, 1996; Dems and Turant, 1997; Bojczuk and Mróz, 1998; Mróz and Bojczuk, 2003; Wang, 2004, 2006; Jang et al., 2009). But beam topology optimization problems concerning locations of supports relative to the ends of bars (hinges) have not yet been resolved completely. Previous articles of the author concerned only a part of the issue – topology and geometry optimization of statically determinate beams under fixed load of different distributions (Rychter and Kozikowska, 2009; Kozikowska, 2011) and under the most unfavorably distributed load (Kozikowska, 2014).

Beam loads are generally a combination of dead and live loads. Dead load is essentially constant and can be treated as uniformly distributed, especially for beams with constant cross-sections. Live load can vary during the life of the structure. In the paper both loads occur simultaneously. It is assumed that live load is characterized by a relatively slow increase of the magnitude and it is regarded as static (without dynamic effects).

Beams are usually subjected to transverse loadings, which result in internal shear forces and bending moments. In the article we consider beams which are relatively long in comparison with their thickness and depth. Bending stresses have the greatest effect on the behavior of such beams and they can be designed mainly against bending moment resistance. Therefore the structural measure of beams is defined in the paper as the absolute maximum bending moment, like in Wang (2006). For beams with uniform cross-sections this measure corresponds to the design for minimum weight. The most adverse distributions of the live load for all cross-sections of a beam can be obtained with the help of influence lines for bending moment.

Due to the complexity of the geometric search space of statically determinate beam with any number of supports, geometry optimization of beams is performed using a genetic algorithm. This method of probabilistic optimization have been applied to a great variety of structural optimization problems (Wang and Chen, 1996; Castro and Partridge, 2006; Rychter and Kozikowska, 2009).

Results of topology optimization, occurring in the literature, usually depend on an initial layout, which is adopted arbitrarily. The final solutions are then obtained by exploring only some parts of the full search space and they are not necessarily the best topological layouts. In the paper the space of all possible beam topologies is known, exhaustive search in this space is carried out and global optima are determined. Moreover, the paper presents not only globally optimal beam topologies, but classifies all topologies into equivalence classes with equal minimum values of the absolute maximum moment. Typical features of these classes are discussed.

2 BEAM TOPOLOGY AND GEOMETRY

The subject of the paper is the space of all statically determinate beams with different topologies, with two or more pin supports. The construction of all possible topologies of beams with n pin supports, solved by Rychter (Rychter and Kozikowska, 2009), starts with the topology, where all n ends of all bars are supported. Then each support can be shifted from the end of a bar (first and last support), or the common hinged end of two adjacent bars (intermediate supports), into the interior of a bar, but not to a distant bar. The topology $\mathbf{t}_i$ of an n-support statically determinate beam is represented by n topological codes of supports t_i:

$$t_i \in \begin{cases} \{0,2\} & \text{for } i=1, \\ \{0,1,2\} & \text{for } i \in \{2,\dots,n-1\}, \\ \{0,1\} & \text{for } i=n, \end{cases} \tag{1}$$

where t_i is equal to 0 for no shift of support i, 1 for shift of support i to the left, and 2 for shift to the right. For example, the beam in Fig. 1 has the topology [2,2,1,1,1,0,...,0,2,1].

The geometry of a beam is described by two sets of geometric parameters: z_i and y_i. The parameters z_i are dimensionless lengths of spans between neighbour supports:

$$0 < z_i < 1, \quad i \in \{1, 2, ..., n-1\} \tag{2}$$

The parameters y_i represent dimensionless lengths of external and internal cantilevers:

$$\begin{aligned} & y_i = 0 \quad \text{if } t_i = 0, \quad i \in \{1, 2, ..., n\} \\ & 0 < y_i < 1 \quad \text{if } t_i \neq 0, \quad i \in \{1, 2, ..., n\} \\ & y_i + y_{i+1} < 1 \quad \text{if } t_i = 1 \wedge t_{i+1} = 2, \quad i \in \{2, ..., n-2\} \end{aligned} \tag{3}$$

When support i is at the end of the beam or at the hinge, no cantilever is created, and the parameter y_i equals zero – the first row in Eq. (3). Otherwise, the parameter y_i takes real value from the interval (0,1) – the second row in Eq. (3). The third row in Eq. (3) prevents the cantilevers from overlapping. For the external cantilevers the parameters y_1, y_n are dimensionless lengths. For the internal cantilevers the parameters $y_2, ..., y_{n-1}$ are ratios of lengths of cantilevers to the lengths of spans in which the cantilevers reside.

The total length of a beam is the sum of the lengths of all spans and the lengths of the external cantilevers. All beams have the same length, normalized to unity:

$$L = y_1 + z_1 + z_2 + ... + z_{n-1} + y_n = 1 \tag{4}$$

A more detailed description of the topological and geometrical parameters is given in Rychter and Kozikowska (2009).

3 GEOMETRY OPTIMIZATION OF A BEAM WITH A FIXED TOPOLOGY

3.1 Problem Formulation

Beams are mainly used in flooring systems of buildings and bridges. In most of these applications beams are prismatic (straight with uniform cross-section) and loaded perpendicularly to the longitudinal direction. Loads of the beams can be categorized into two groups: dead (fixed) loads and live (temporary) loads. Dead loads are gravity loads due to the self-weight of the beams and all other material and equipment permanently attached to them. The magnitude and spatial distribution of the dead loading are constant over time. Dead load is sometimes the most important part of the beam loading, particularly for beams with long spans and built of heavy materials. In some cases the importance of this load can be reduced in relation to other loads, but this load should not be ignored. For prismatic beams this load is mainly uniformly distributed and this case is considered in the article. Live loads are usually gravity-type (possibly piecewise) loads of regularly or irregularly varying magnitudes and/or varying positions caused by the use of the structure. Examples of temporary loads are stored items, furniture, occupants in buildings and pedestrians on footbridges. Although the loads are movable it is assumed in the paper that they are applied slowly and there is no dynamic amplification. In such a case, these moveable loads are considered as quasi-static arbitrarily distributed loads. Because of the variability of the load, we have to consider all possible live load combinations and find the ones that result in the maximum values of bending moment. The issues discussed in the article do not depend on the absolute values of the dead and live load intensities, but only on their ratio. Therefore we normalize both intensities so that their sum is constant, equal to one. The intensity of dead load is equal to q and the maximum intensity of live load is equal to $1-q$. Each intensity can take values from 0 to 1.

Beams under arbitrarily distributed transverse live loads, considered in the author's article (Kozikowska, 2014), had two most unfavorable load cases for the maximum bending moment. Each case included uniformly distributed load of maximum intensity on alternate spans. If we take dead and live load into account, we also have two load cases. One of the cases comprises uniform load of intensity 1 (sum of both loads) on odd spans and uniform dead load of intensity q on even spans. The second case also includes uniform loads: q on odd spans and sum of loads equal to 1 on even spans.

We assume that each beam is of unit length with a fixed topology $\mathbf{t}_i$. The geometry optimization problem is defined as follows:

$$\text{Minimize } \max_{x\in[0,1]} |M(z_i, y_j, x)| \tag{5}$$

$$\text{Subject to } \begin{cases} 0 < z_i < 1 \quad i = 1,2,\dots,n-1 \\ 0 < y_j < 1 \quad \text{for } t_j \neq 0 \quad j = 1,2,\dots,n \\ y_1 + z_1 + z_2 + \dots + z_{n-1} + y_n = 1 \end{cases} \tag{6}$$

where $\max_{x\in[0,1]} |M(z_i, y_j, x)|$ denotes the maximum of the absolute bending moment (objective function) for both load cases, z_i are span lengths given by Eq. (2), y_j are nonzero lengths of cantilevers, which are created by shifts of supports of nonzero topological codes t_j from ends of bars, given by Eq. (3), and x is the axial coordinate.

This geometry optimization has been carried out by a modified version of the genetic algorithm (Rychter and Kozikowska, 2009), written by the author in C/C++ programming language.

3.2 Genetic Algorithm

The genetic algorithm for the optimization of the geometry of statically determinate beams follows the general scheme of genetic algorithms (Goldberg, 1989). Populations of chromosomes are evolved over several generations, subject to random mutation, random crossover (recombination), and selection pressure.

The chromosome $\mathbf{g}$ representing a geometry of a n-support statically determinate beam with a fixed topology $\mathbf{t}_i$ is a string of $n-1$ real genes z_i and $(c_E + c_H)$ real nonzero genes y_j:

$$\mathbf{g} = [z_1, \dots, z_{n-1}, \dots, y_j, \dots] \tag{7}$$

where z_i, y_j are given by Eq. (2), (3), and (4).

The creation of an initial population involves the assignment of random real values from the interval (0,1) to all genes, and then the adjustment of the genes to the conditions contained in the third row of Eq. (3) and in Eq. (4).

The designed chromosomes allow for easy random mutation and crossover, without producing incorrect beams. After these operations, the chromosomes only have to be adjusted to the conditions given in the third row of Eq. (3) and in Eq. (4). The most efficient versions of mutation, crossover and selection have been determined through extensive simulations.

Gaussian mutation, in which a Gaussian distributed random value is added to the value of the chosen gene, has turned out to be the best mutation method.

The performance of algorithm with single-point crossover was the same as with multi-point crossover. Yet the former is simpler and faster than the latter. Therefore, single-point recombination has been used where two parent chromosomes are cut in one random point, and both chromosome parts are swapped to produce two children.

Three selection strategies have been studied: proportional roulette-wheel, proportional deterministic, and ranking tournament. The tournament selection involves running several tournaments (groups of chromosomes) chosen at random from population members. The winners of all tournaments go into the next population. Moreover, in the applied tournament strategy, the best beam must fall into at least one tournament group and so will always survive selection. Numerical simulations have shown the superiority of this tournament selection with binary tournaments over proportional selection methods.

The minimal value of the absolute maximum bending moment M_i has been found for each topology $\mathbf{t}_i$ as a result of the optimization by this genetic algorithm.

3.3 Optimization Results

A beam with optimal geometry for the fixed topology is presented in Fig. 1. The beam is shown with two unique bending moment diagrams, drawn with a solid line or a dashed line, for both the most unfavorable load cases. The optimal envelope of the two moment diagrams has the same local extreme moment values equal to M_i. These values are present over the supports which were moved away from ends of bars and at the bottom at mid spans or close to them. The envelope has zero values exclusively in hinges and at both ends of the beam and is equivalent to only one topology, unlike in the case of dead load alone (Kozikowska, 2011).

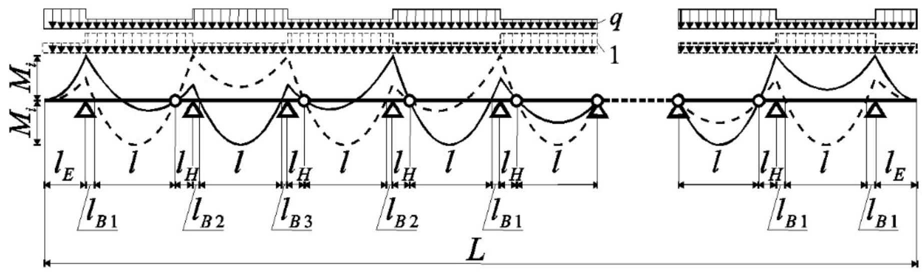

Figure 1: A beam with optimal geometry for the fixed topology [2,2,1,1,1,0,...,0,2,1].

We are interested in analytical expressions for optimal geometrical parameters for any topology, under stationary load and the most unfavorably distributed load. In order to determine the values of the parameters l, l_E, l_H, and l_{Bj} for $j = 1, 2, \ldots, n-1$ (see Fig. 1) we solve the system of equations:

$$(n-1)l + c_E l_E + c_H l_H + \sum_{j=1}^{n-1} c_{Bj} l_{Bj} = L \tag{8}$$

$$\frac{1}{2}l - l_E = 0 \tag{9}$$

$$l^2 - 4l l_H - 4(l_H)^2 = 0 \tag{10}$$

$$(l + l_{B1})l_{B1} - q(l + l_H)l_H = 0 \tag{11}$$

$$(l + l_{B(j-1)})\left((l + l_{Bj})l_{Bj} - q(l + l_H + l_{B(j-1)})l_H\right) + \frac{1}{4}l^2 l_H = 0 \tag{12}$$

where l_E and l_H denote the lengths of nonzero external and internal cantilevers, respectively, l is the length of each segment with at least one of the two optimal moment diagrams at the bottom, with the maximum value of this moment equal to M_i and zero values of this moment at both ends of the segment, l_{Bj} for $j = 1, 2, \ldots n-1$ is the distance from intersection of optimal moment diagram (with maximum value M_i at the bottom) with beam axis to the nearest support, moreover there is no hinge in zero moment point. The lengths l_{Bj} neighbour on the external and internal cantilevers. The indices j in l_{Bj} are consecutive numbers of these neighbouring cantilevers, counted from the top of the interaction scheme of the beam (see Fig. 2). The neighbouring cantilevers, which form consecutive levels (steps) in the interaction diagram, create sequences. The number of terms (cantilevers) in such a sequence is the length of the cantilever sequence. The pseudo code for an algorithm to assign the lengths l_{Bj} to supports on the basis of the beam topology is given in appendix A. The algorithm returns a vector $\mathbf{b}_i$ of n integer elements b_k. The element b_k is equal to j if the length l_{Bj} is on the right side of the support k, is equal to $-j$ if the length l_{Bj} is on the left side of the support k, and is equal to zero if there is no length l_{Bj} next to support k (the support k is at the end of the beam or under a hinge). For example, $\mathbf{b}_i = [1, 2, -3, -2, -1, 0, \ldots, 0, 1, -1]$ for the beam from Fig. 2.

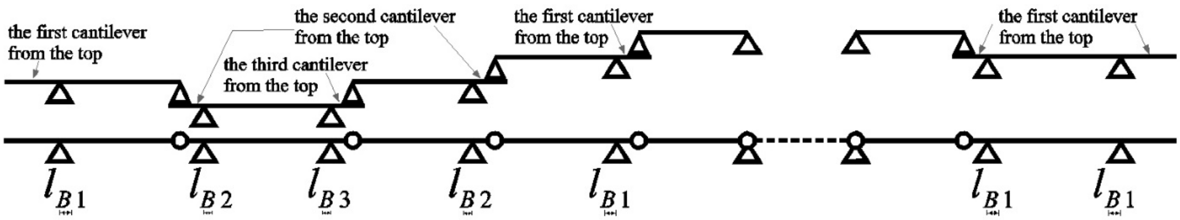

Figure 2: Locations of lengths l_{Bj} depending on consecutive numbers of cantilevers in the interaction diagram of the beam.

Equation (8) describes the length of the beam as the sum of individual segment lengths. The parameters c_E, c_H, c_{Bj} for $j = 1, 2, \ldots n-1$ are the numbers of the segments l_E, l_H, l_{Bj} for $j = 1, 2, \ldots n-1$, respectively. The lengths of a cantilever and a simply supported beam with the same values of the absolute maximum moment under uniformly distributed dead and live load are compared in Eq. (9). The maximum bending moment value of a simply supported beam of the length $l + 2l_H$ equals twice this moment value of a simply supported beam of the length l in accordance with Eq. (10). Eq. (11) is explained graphically in Fig. 3. The equation was established by comparing the moment value at the support calculated on the basis of the moment diagram on the left of the support with this moment value calculated according to the moment diagram on the right. Eq. (12) was found from the moment diagram, drawn with a solid line in Fig. 4.

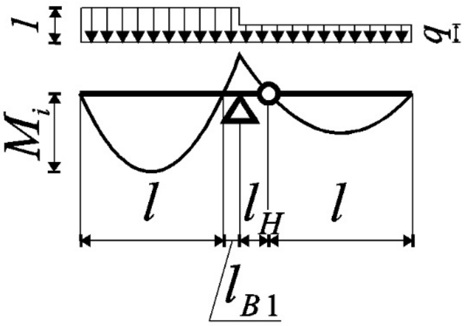

Figure 3: Graphic explanation of Eq. (11).

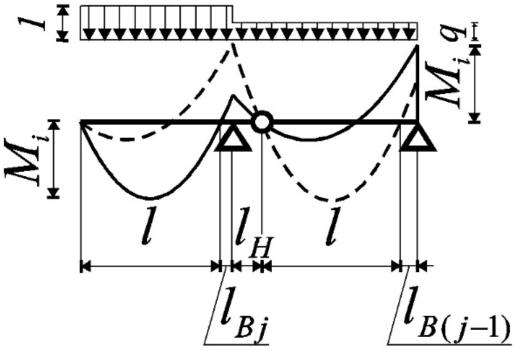

Figure 4: Graphic explanation of Eq. (12).

The solution to the system of equations (8)–(12) is given by:

$$l = \frac{L}{d} \tag{13}$$

$$l_E = \frac{L}{2d} \tag{14}$$

$$l_H = \left(\sqrt{2}-1\right)\frac{L}{2d} \tag{15}$$

$$l_{Bj} = \left(B_j - 1\right)\frac{L}{2d} \quad \text{for } j = 1,\ldots,\ n-1 \tag{16}$$

where

$$d = n + \frac{1}{2}c_E + \frac{1}{2}(\sqrt{2}-1)c_H + \frac{1}{2}\sum_{j=1}^{n-1}\left[\left(B_j - 1\right)c_{Bj}\right] - 1$$

$$B_1 = \sqrt{q+1}$$

$$B_j = \sqrt{1 - \frac{\sqrt{2}-1}{B_{(j-1)}+1} + q\left(\sqrt{2}-1\right)\left(B_{(j-1)} + \sqrt{2}\right)} \quad \text{for } j = 2,\ldots,\ n\text{-}1$$

The values of the parameters c_E and c_H can be determined from the beam topology $\mathbf{t}_i$. The value of the parameter c_E is equal to the number of nonzero elements in the first and last position of the code $\mathbf{t}_i$. The value of the parameter c_H is the number of nonzero elements in positions 2 through $n-1$ of the code $\mathbf{t}_i$. An algorithm to calculate the values of parameters c_{Bj} for $j = 1, 2, \ldots n-1$ on the basis of the vector $\mathbf{b}_i$ (assigning the lengths l_{Bj} to supports) is given by a pseudo code in appendix B. The value of the absolute maximum bending moment M_i can be calculated as the moment in the centre of a simply supported beam of the length l under uniform load equal to the sum of both loads (intensity equal to one):

$$M_i = l^2/8 \tag{17}$$

Algorithms to calculate the coordinates of supports and hinges (on the basis of the beam topology, the vector $\mathbf{b}_i$, and the lengths l, l_E, l_H, and l_{Bj} for $j = 1, 2, \ldots, n-1$) are given by a pseudo code in appendix C.

3.4 Dependence of Optimal Geometrical Parameters on Dimensionless Dead Load Intensity q

The formulas (13)–(16) enable us to calculate the optimal lengths of the segments l, l_E, l_H, l_{Bj} for $j = 1, 2, \ldots n-1$ for any number of supports, for any topology, and for any value of the dimensionless dead load intensity $0 \le q \le 1$. For the extreme values of q, equal to 0 or 1, we receive special cases with specific values of l_{Bj}.

For $q = 0$ (only most unfavorably distributed load) regardless of the beam topology, the value of the parameter l_{B1} is equal to 0, and the values of the parameters l_{Bj} for $j = 2, \ldots n-1$, calculated from the formula (16), are less than 0. The negative value of the parameter l_{Bj} means that the segment

l_{Bj} is on the same side of the support as the segment l_H (see Fig. 5a for the beam with the topology [2,2,1,1]).

For $q=1$ (only fixed load) regardless of the beam topology, both load cases come down to one case with dead load on the entire beam, and all segments l_{Bj} for $j=1,\ldots n-1$ have the same length equal to the length l_H (see Fig. 5c for the beam with the topology [2,2,1,1]).

For $0<q<1$ (fixed load and most unfavorably distributed load acting simultaneously) regardless of the beam topology, all optimal segment lengths have different values. The segments l_{Bj} are shorter than l_H, while l, l_E are longer than l_H. The value of the parameter l_{B1} is more than zero, but for small values of q (less than about 0.2) the lengths l_{Bj} for $j=2,\ldots n-1$ are less than zero.

The lengths of the optimal beam segments can also be calculated in case of live load alone using the formulas presented in Kozikowska (2014) and in case of dead load alone using the formulas given in Kozikowska (2011). The sets of optimal geometrical parameters which have been used in the previous author's articles consist of a smaller number of parameters than the set used in this paper. Therefore, the sets used in Kozikowska (2014, 2011) cannot be applied to describe the optimal geometry of the statically determinate beams for $0<q<1$.

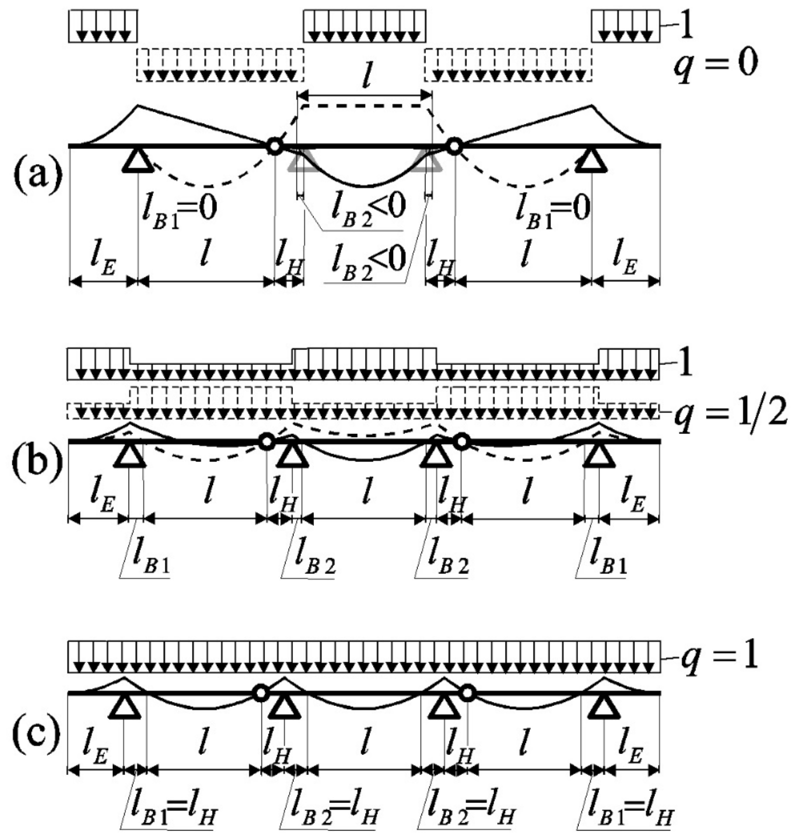

Figure 5: The beam with optimal geometry for the topology [2,2,1,1] and for different values of the dimensionless dead load intensity q: (a) $q=0$, (b) $q=1/2$, (c) $q=1$.

The dependence of optimal segment lengths on values of q is illustrated in Fig. 6 with regard to the beam from Fig. 5 (with the topology [2,2,1,1]). The values of the parameters l, l_E, l_H are more than zero for all values of $0 \le q \le 1$, and the values l, l_E, l_H are the biggest for $q=0$. For $q=0$, the value of the parameter l_{B1} is equal to zero, but l_{B2} is less than zero. Next, the values of the parameters l, l_E, l_H decrease with increasing q, l_{B1} and l_{B2} increase with increasing q, and l_{B1} and l_{B2} reach the value l_H for $q=1$. For q greater than zero but less than a value of about 0.2, the value of the parameter l_{B2} is less than 0. The dependence of the parameters l, l_E, l_H, l_{Bj} for $j=1,2,\ldots n-1$ on q is the same for beams with any number of supports and any topology.

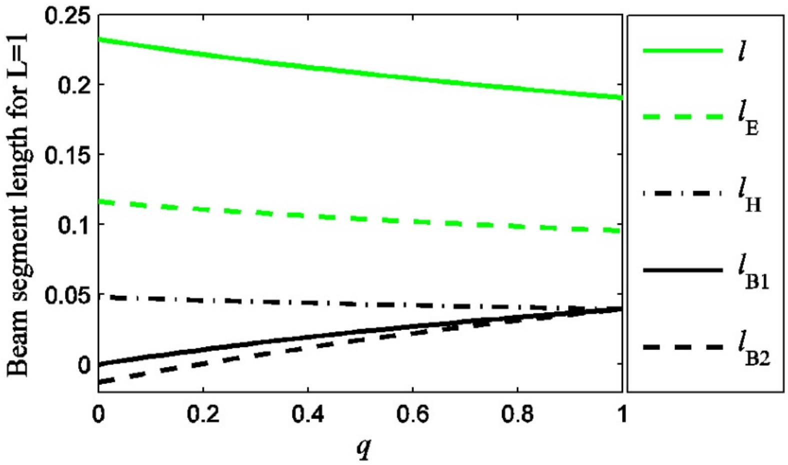

Figure 6: Lengths l, l_E, l_H, l_{B1}, and l_{B2} of the beam from Fig. 5 (with the topology [2,2,1,1]) for different values of the dimensionless dead load intensity q.

4 TOPOLOGY OPTIMIZATION OF BEAMS WITH A FIXED NUMBER OF SUPPORTS FOR $0 < q < 1$

Topology optimization of beams with a fixed number of supports for $q=0$ (live load alone) is presented in Kozikowska (2014) and for $q=1$ (dead load alone) in Kozikowska (2011).

4.1 Equivalence Relation of Beam Topologies

$\mathbf{T}$ is the set of n-support beam topologies: $\mathbf{T}^n$ or $\mathbf{T}^{2:n}$. Any two topologies $\mathbf{t}_i$ and $\mathbf{t}_j$ of the set $\mathbf{T}$ are equivalent with respect to the relation R if the minimal values of the absolute maximum moments M_i and M_j of these topologies are equal:

$$\mathbf{t}_i \equiv_R \mathbf{t}_j \quad \text{if } M_i = M_j \tag{18}$$

Based on this relation R, the set $\mathbf{T}^n$ can be divided into disjoint equivalence classes of beam topologies called topological classes $\mathbf{T}_i^n$, and the set $\mathbf{T}^{2:n}$ into topological classes $\mathbf{T}_i^{2:n}$.

4.2 Features of Beam Topologies in a Topological Class

All optimal bending moment diagram pairs from the topological class $\mathbf{T}_{18}^{6}$ (the eighteenth class of all six-support classes ordered by increasing values of moments M_i^n), under a fixed uniformly distributed load and the most unfavorably piece-wisely distributed load, are shown in Fig. 7.

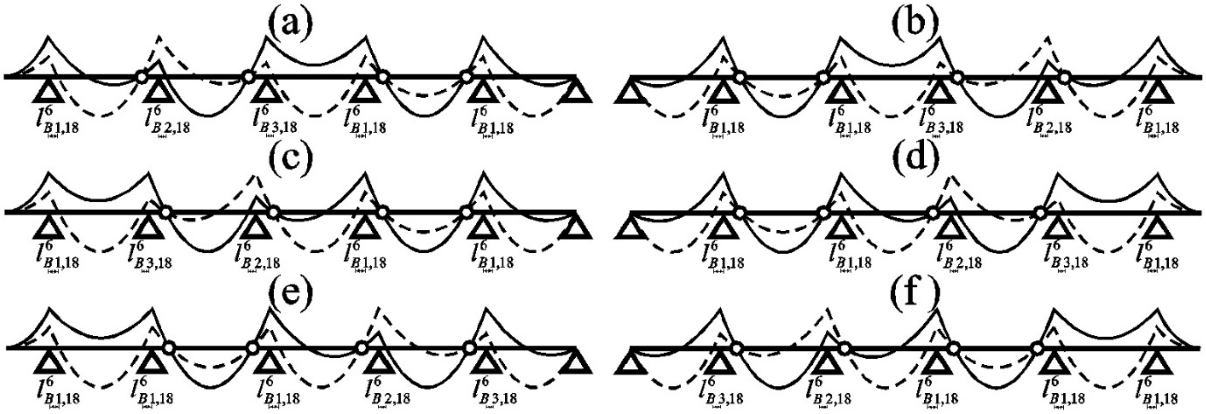

Figure 7: All optimal envelopes of moment diagrams in the class $\mathbf{T}_{18}^{6}$: $c_{E,18}^{6}=1$, $c_{H,18}^{6}=4$, $c_{B1,18}^{6}=3$, $c_{B2,18}^{6}=1$, $c_{B3,18}^{6}=1$, $c_{B4,18}^{6}=0$, $c_{B5,18}^{6}=0$.

All topologies in the topological class $\mathbf{T}_i^n$ have the same values of moment M_i^n and lengths l_i^n, $l_{E,i}^n$, $l_{H,i}^n$, $l_{Bk,i}^n$ for $k=1,2,\ldots n-1$. The lengths l_i^n, $l_{E,i}^n$, $l_{H,i}^n$, $l_{Bk,i}^n$ for $k=1,2,\ldots n-1$, given by Eq. (13)–(16), depend on the number of supports, the values of the paraml1eters $c_{E,i}^n$, $c_{H,i}^n$, and $c_{Bk,i}^n$ for $k=1,2,\ldots n-1$, and the value of q. Thus for two topologies $\mathbf{t}_i$ and $\mathbf{t}_j$ of the set $\mathbf{T}^n$ under a fixed and the most unfavorably distributed load the equivalent condition from Eq. (18) can be expressed as:

$$\mathbf{t}_i \equiv_R \mathbf{t}_j \quad \text{if } c_{E,i}=c_{E,j} \;\wedge\; c_{H,i}=c_{H,j} \;\wedge\; c_{Bk,i}=c_{Bk,j} \text{ for } k=1,\ldots,n-1 \tag{19}$$

where $c_{E,i}$, $c_{H,i}$, $c_{Bk,i}$ for $k=1,2,\ldots n-1$, $c_{E,j}$, $c_{H,j}$, $c_{Bk,j}$ for $k=1,2,\ldots n-1$ are the numbers of the appropriate segments for the topology $\mathbf{t}_i$ and $\mathbf{t}_j$, respectively.

4.3 Comparison of Topological Classes

The whole sets of topological classes under a fixed uniform and the most unfavorably distributed load, with all optimal envelopes of moment diagrams are presented in Fig. 8 (for three support and equal intensities of dead and live loads, $q=1/2$), Fig. 9 (for three support and live load intensity eight times greater than dead load intensity, $q=1/9$), and Fig. 10 (for four support and equal intensities of dead and live loads, $q=1/2$).

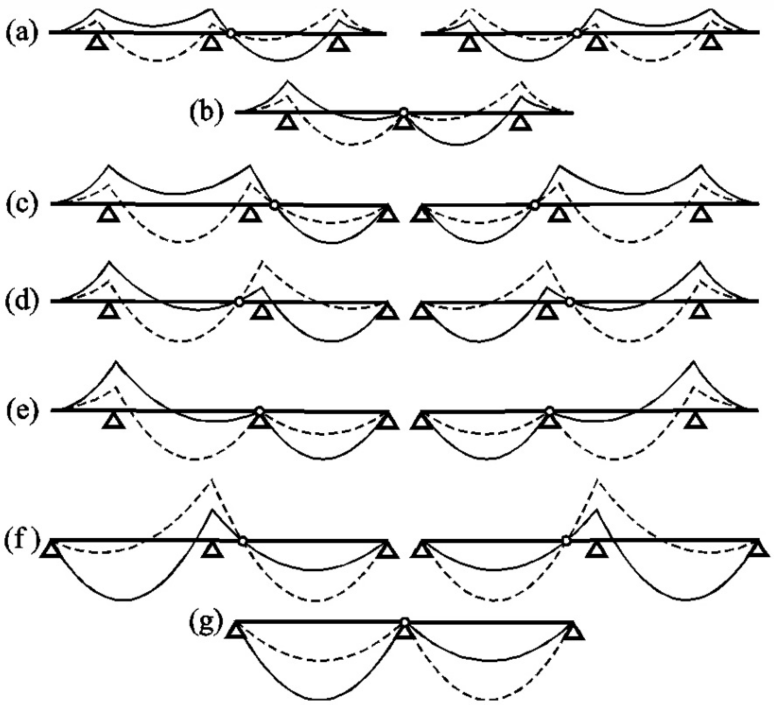

Figure 8: All three-support topological classes with their optimal envelopes of moment diagrams: (a) $\mathbf{T}_1^3$, (b) $\mathbf{T}_2^3$, (c) $\mathbf{T}_3^3$, (d) $\mathbf{T}_4^3$, (e) $\mathbf{T}_5^3$, (f) $\mathbf{T}_6^3$, (g) $\mathbf{T}_7^3$ ($q = 1/2$).

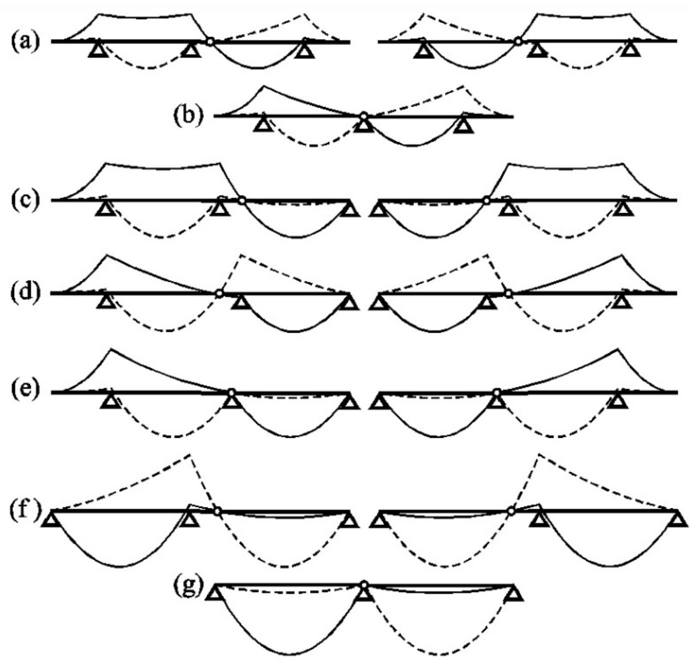

Figure 9: All three-support topological classes with their optimal envelopes of moment diagrams: (a) $\mathbf{T}_1^3$, (b) $\mathbf{T}_2^3$, (c) $\mathbf{T}_3^3$, (d) $\mathbf{T}_4^3$, (e) $\mathbf{T}_5^3$, (f) $\mathbf{T}_6^3$, (g) $\mathbf{T}_7^3$ ($q = 1/9$).

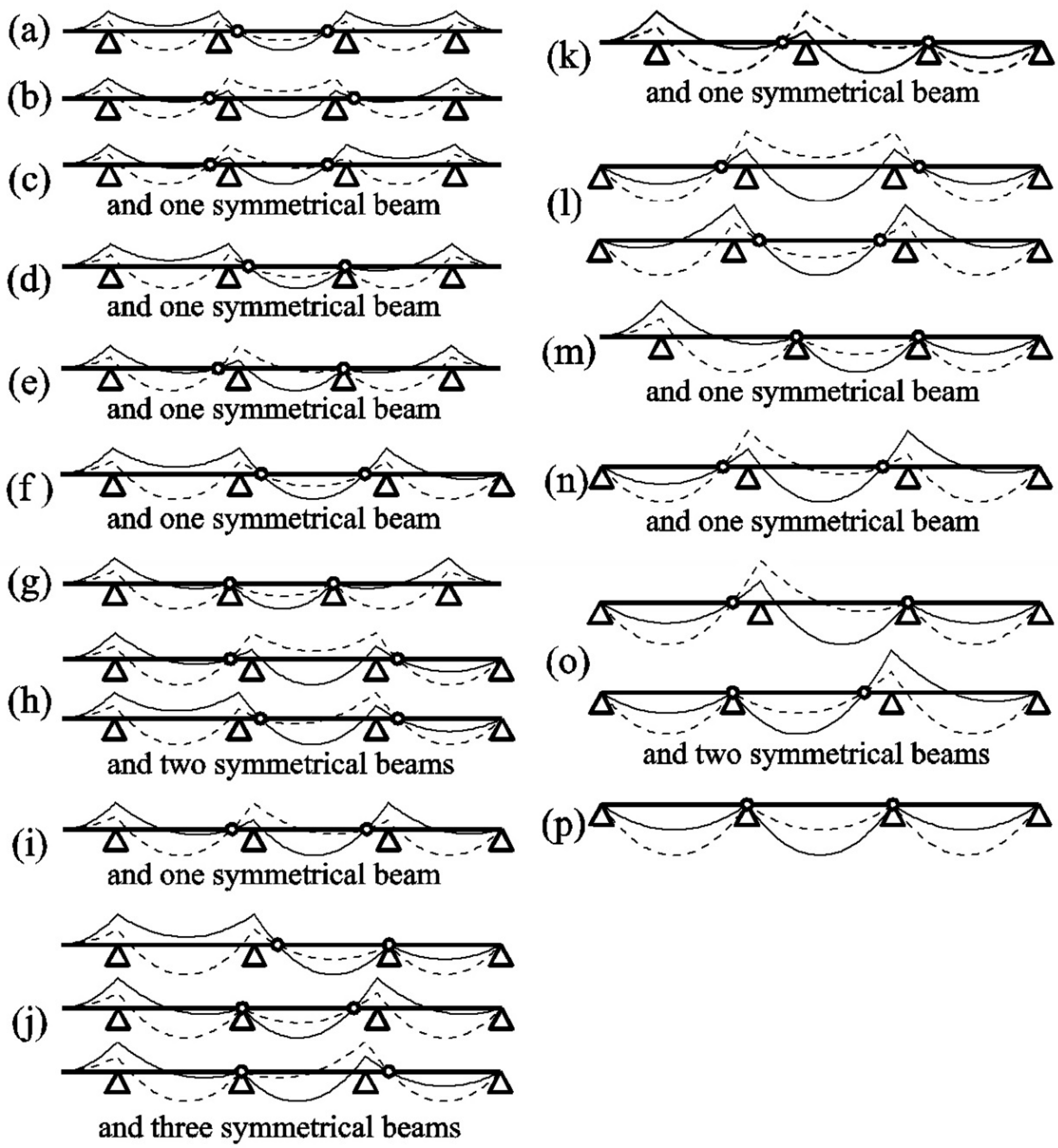

Figure 10: All four-support topological classes with their optimal envelopes of moment diagrams: (a) $\mathbf{T}_1^4$, (b) $\mathbf{T}_2^4$, (c) $\mathbf{T}_3^4$, ... , (p) $\mathbf{T}_{16}^4$ ($q = 1/2$).

The division of beam topologies into topological classes does not depend on the value of the dead to live load ratio. This ratio only affects the optimal values of the geometrical parameters, which can be calculated from the formulas (13)–(16). The lengths $l_{E,i}^n$, $l_{H,i}^n$, l_i^n, and the moment value M_i^n are greater for smaller values of the ratio (for smaller values of dimensionless dead load intensity q) for all classes except for the last class whose optimal moment is independent of q . The dependence of values M_i^n on q in three-support classes is shown in Fig. 11. It is observed that the growth of q (smaller share of live load) makes moment values decrease, except for the last class $\mathbf{T}_7^3$.

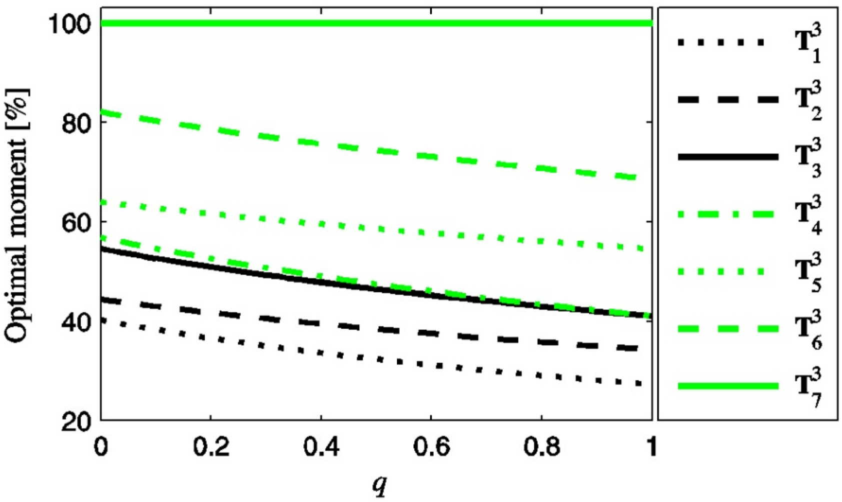

Figure 11: Optimal moments in three-support topological classes for different values of the dimensionless dead load intensity q .

The division of all topologies into topological classes depends on the number of cantilevers and their locations in the interaction diagram of the beam (see Fig.2). The topological classes are the better, the more cantilevers their beams have (the more external cantilevers for the same total number of cantilevers) and the shorter the lengths of the cantilever sequences are in the interaction schemes. In other words, better classes have larger values of the parameters $c_{E,i}^n$ and $c_{H,i}^n$ (have larger values of the parameters $c_{E,i}^n$ than the parameters $c_{H,i}^n$ for the same sum of $c_{E,i}^n$ and $c_{H,i}^n$), and have more zero parameters $c_{Bj,i}^n$. The values of the parameters $c_{E,i}^n$, $c_{H,i}^n$, and $c_{Bj,i}^n$ for $j=1,\ldots n-1$ for the three-support classes (from Fig. 8 and Fig. 9) and for the four-support classes (from Fig. 10) are given in Table 1 and Table 2, respectively. The best topological class with an odd number of supports have $n-1$ topologies, each with a single one-hinged span (see Fig. 8a and Fig. 9a). The best single topology in the first class with an even number of supports does not have any one-hinged spans (see Fig. 10a).

$\mathbf{T}_i^3$	$\mathbf{T}_1^3$	$\mathbf{T}_2^3$	$\mathbf{T}_3^3$	$\mathbf{T}_4^3$	$\mathbf{T}_5^3$	$\mathbf{T}_6^3$	$\mathbf{T}_7^3$
$c_{E,i}^3$	2	2	1	1	1	0	0
$c_{H,i}^3$	1	0	1	1	0	1	0
$c_{B1,i}^3$	2	2	2	1	1	1	0
$c_{B2,i}^3$	1	0	0	1	0	0	0

Table 1: Values of the parameters $c_{E,i}^3$, $c_{H,i}^3$, $c_{Bj,i}^3$ for $j=1,2$ in three-support topological classes.

$\mathbf{T}_i^4$	$\mathbf{T}_1^4$	$\mathbf{T}_2^4$	$\mathbf{T}_3^4$	$\mathbf{T}_4^4$	$\mathbf{T}_5^4$	$\mathbf{T}_6^4$	$\mathbf{T}_7^4$	$\mathbf{T}_8^4$	$\mathbf{T}_9^4$	$\mathbf{T}_{10}^4$	$\mathbf{T}_{11}^4$	$\mathbf{T}_{12}^4$	$\mathbf{T}_{13}^4$	$\mathbf{T}_{14}^4$	$\mathbf{T}_{15}^4$	$\mathbf{T}_{16}^4$
$c_{E,i}^4$	2	2	2	2	2	1	2	1	1	1	1	0	1	0	0	0
$c_{H,i}^4$	2	2	2	1	1	2	0	2	2	1	1	2	0	2	1	0
$c_{B1,i}^4$	4	2	2	3	2	3	2	2	1	2	1	2	1	1	1	0
$c_{B2,i}^4$	0	2	1	0	1	0	0	1	1	0	1	0	0	1	0	0
$c_{B3,i}^4$	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0

Table 2: Values of the parameters $c_{E,i}^4$, $c_{H,i}^4$ and $c_{Bj,i}^4$ for $j=1,2,3$ in four-support topological classes.

The set of all n-support topological classes is described by the set of all possible $(n+1)$-element sequences $(c_{E,i}^n, c_{H,i}^n, c_{Bj,i}^n \text{ for } j=1,2,\ldots n-1)$ where $c_{E,i}^n \in \{0,1,2\}$, $c_{H,i}^n \in \{0,1,\ldots,n-2\}$, and values of the parameters $c_{Bj,i}^n$ for $j=1,2,\ldots n-1$ meet the following conditions:

$$\forall j \in \{1,2,\ldots,n-1\}\; c_{Bj,i}^n \geq 0 \tag{20}$$

$$c_{B1,i}^n \geq c_{E,i}^n \tag{21}$$

$$\forall j \in \{1,2,\ldots,n-2\}\; c_{Bj,i}^n \geq c_{Bj+1,i}^n \tag{22}$$

$$\left(c_{E,i}^n = 2 \wedge c_{H,i}^n = n-2\right) \Rightarrow c_{B1,i}^n \text{ is even} \tag{23}$$

$$\sum_{j=1}^{n-1} c_{Bj,i}^n = c_{E,i}^n + c_{H,i}^n \tag{24}$$

The numbers of j-th cantilevers from the top of the interaction scheme $c_{Bj,i}^n$ are nonnegative (see Eq. (20)). External cantilevers are always at the top of the interaction diagram (they are always the first from the top of the interaction scheme) in accordance with Eq. (21). The number of j-th cantilevers from the top of the interaction diagram must be equal to or larger than the number of $(j+1)$-th cantilevers because $(j+1)$-th cantilevers are below j-th cantilevers according to Eq. (22). A bar with two supports is always at the bottom of the interaction diagram. For beams with the maximum number of external and internal cantilevers equal to n, the first cantilever is on both sides of each two-support bar, at the top of a cantilever sequence. Thus, if the number of external and internal cantilevers is maximal, then the number of the first cantilevers $c_{B1,i}^n$ is equal to double the number of two-support bars which means that $c_{B1,i}^n$ is even (see Eq. (23)). Eq. (24) compares the number of cantilevers in the interaction scheme and in the topology. The total number of classes p^n can be calculated by an algorithm that counts the number of the sequences $(c_{E,i}^n, c_{H,i}^n, c_{Bj,i}^n \text{ for } j=1,2,\ldots n-1)$ and is given by a pseudo code in appendix D. The numbers of n-support topological classes for $n \in \{2,3,\ldots,16\}$ are shown in Table 3.

supports	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
topological classes	3	7	16	28	49	78	123	183	272	390	556	774	1072	1459	1977

Table 3: Number of supports vs number of topological classes.

5 TOPOLOGY OPTIMIZATION OF BEAMS WITH A DIFFERENT NUMBER OF SUPPORTS

Assume $\mathbf{T}^{2:n}$ is the set of beam topologies with two to n supports and $\mathbf{T}_i^{2:n}$ is the topological class from this set.

For $q=0$ (only live load) or $q=1$ (only dead load), some classes $\mathbf{T}_i^{2:n}$ contain topologies with two successive numbers of supports. Such a class $\mathbf{T}_i^{2:n}$ is then the sum of k-support class $\mathbf{T}_i^{k}$ and $(k+1)$-support class $\mathbf{T}_i^{k+1}$ for $2 \le k \le n-1$. It happens because the length of parameter l_{B1} is equal to zero for $q=0$, and the lengths of all parameters l_{Bj} for $j=1,\dots n-1$ are equal to l_H for $q=1$. Therefore, the total number of topological classes $p^{2:n}$ in the set $\mathbf{T}^{2:n}$ is for these loads less than the sum of numbers of classes in all sets from $\mathbf{T}^2$ to $\mathbf{T}^n$:

$$p^{2:n} < \sum_{i=2}^{n} p^i \text{ for } q=0 \vee q=1 \tag{25}$$

For $0<q<1$, all classes $\mathbf{T}_i^{2:n}$ consist of topologies with only one number of support. Therefore, the total number of topological classes $p^{2:n}$ in the set $\mathbf{T}^{2:n}$ is for these loads equal to the sum of numbers of classes in all sets from $\mathbf{T}^2$ to $\mathbf{T}^n$:

$$p^{2:n} = \sum_{i=2}^{n} p^i \text{ for } 0<q<1 \tag{26}$$

6 CONCLUSIONS

The paper presents results of geometry and topology optimization of statically determinate beams with an arbitrary number of supports. The beams are exposed to uniform dead load and live load of the most unfavorable distribution. The fixed topology problem involving the geometry optimization for a given topology is solved for each beam. The absolute maximum bending moment is the objective function in this optimization. For this function, it suffices to consider only two load cases, each with dead load on all spans and uniform live load of the maximum possible intensity on alternate spans. Exact formulas for optimal geometrical parameters have been obtained for all topologies and for any dead to live load ratio on the basis of properties of the optimal moment diagram envelopes. Beams of different topologies and equal minimum values of the absolute maximum moment have been assigned to the same topological classes. It has been found that the division of the beam topologies into the topological classes depends on the number of beam cantilevers and their locations in beam interaction diagrams. It has also been found that this division does not depend on the dead to live load ratio for

$0 < q < 1$. Topologies with the maximum number of external and internal cantilevers and with minimal lengths of cantilever sequences in interaction schemes have been found to be the best options.

The article provides some practical guidelines on how to design statically determinate beam structures with the minimum weight. The examination of all topologically different beams provides tremendous design opportunities because it offers a variety of satisfactory solutions, not only the best ones.

Acknowledgements

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References

Bojczuk, D., Mróz Z., (1998). On optimal design of supports in beam and frame structures. Structural Optimization 16: 47–57.

Castro, L.L.B., Partridge, P.W., (2006). Minimum weight design of framed structures using a genetic algorithm considering dynamic analysis. Latin American Journal of Solids and Structures 3: 107–123.

Dems, K., Turant, J., (1997). Sensitivity Analysis and Optimal Design of Elastic Hinges and Supports in Beam and Frame Structures. Mechanics of Structures and Machines 25: 4, 417–443.

Eschenauer, H.A., Olhoff, N., (2001). Topology optimization of continuum structures: A Review. Applied Mechanics Review: 54, 331-390.

Fancello, E.A., Pereira, J.T., (2003). Structural topology optimization considering material failure constraints and multiple load conditions. Latin American Journal of Solids and Structures 1: 3-24.

Goldberg, D.E., (1989). Genetic algorithms in search, optimization and machine learning, Addison-Wesley: Reading Mass.

Imam, M.H., Al-Shihri M., (1996). Optimum topology of structural supports. Computers and Structures 61: 147–154.

Jang, G.W., Shim, H.S., Kim, Y.Y. (2009). Optimization of support locations of beam and plate structures under self-weight by using a sprung structure model. Journal of Mechanical Design 131, 2, 021005.1-021005.11.

Kirsch, U., (1989). Optimal topologies of structures. Applied Mechanics Reviews 42: 8, 223–239.

Kozikowska, A., (2011). Topological classes of statically determinate beams with arbitrary number of supports. Journal of Theoretical and Applied Mechanics 49: 4, 1079–1100.

Kozikowska, A., (2014). Topological classes of statically determinate beams with arbitrary number of supports under the most unfavourably distributed load. Journal of Theoretical and Applied Mechanics 52: 1, 257–269.

Lopes, C.G., Batista dos Santos, R., Novotny, A.A., (2015). Topological derivative-based topology optimization of structures subject to multiple load-cases. Latin American Journal of Solids and Structures 12: 834–860.

Marczak, R.J., (2008). Optimization of elastic structures using boundary elements and a topological-shape sensitivity formulation. Latin American Journal of Solids and Structures 5: 99–117.

Mróz, Z., Bojczuk, D., (2003). Finite topology variations in optimal design of structures. Structural and Multidisciplinary Optimization 25: 153–173.

Mróz, Z., Rozvany, G.I.N., (1975). Optimal design of structures with variable support conditions. Journal of Optimization Theory and Applications 15: 85–101.

Rozvany, G.I.N., (2009). A critical review of established methods of structural topology optimization. Structural and Multidisciplinary Optimization 37: 217–237.

Rozvany, G.I.N., Bendsøe, M.P., Kirsch, U., (1995). Layout optimization of structures. Applied Mechanics Reviews 48: 41–119.

Rychter, Z., Kozikowska, A., (2009). Genetic algorithm for topology optimization of statically determinate beams. Archives of Civil Engineering 55: 1, 103–123.

Timoshenko, S.P., (1953). History of Strength of Materials, McGraw-Hill Book Company, Inc. (New York).

Wang, B.P., Chen, J.L., (1996). Application of genetic algorithm for the support location optimization of beams. Computers and Structures 58: 797–800.

Wang, D., (2004). Optimization of support positions to minimize the maximal deflection of structures. International Journal of Solids and Structures 41: 7445–7458.

Wang, D., (2006). Optimal design of structural support positions for minimizing maximal bending moment. Finite Elements in Analysis and Design 43: 95–102.

Appendix A - Pseudo Code for the Algorithm to Assign the Lengths l_{Bj} to Supports for Beam Topology

FUNCTION assigning_lengths_lB_to_supports()
INPUT: the topological code of beam: n-element vector $\mathbf{t}$
OUTPUT: n parameters assigning lengths l_{Bj} to supports: n-element vector $\mathbf{b}$
FOR i starts at 1, $i \leq n$, increment i DO ASSIGN to b_i the value of 0
END FOR
(* assigning lengths lB to supports for topological codes 2 *)
ASSIGN to i the value of 1
WHILE i is less than or equal to n DO
    IF t_i is equal to 2 THEN
        ASSIGN to b_i the value of 1
        WHILE (i is less than n) and (t_{i+1} is equal to 2) DO
            ADD 1 to i
            ASSIGN to b_i the value of $b_{i-1}+1$
        END WHILE
        ADD 1 to i
    ELSE ADD 1 to i
    END IF
END WHILE
(* assigning lengths lB to supports for topological codes 1 *)
ASSIGN to i the value of n
WHILE i is greater than or equal to 1 DO
    IF t_i is equal to 1 THEN
        ASSIGN to b_i the value of -1
        WHILE (i is greater than 1) and (t_{i-1} is equal to 1) DO
            SUBTRACT 1 from i
                ASSIGN to b_i the value of $b_{i+1}-1$
        END WHILE
        SUBTRACT 1 from i
    ELSE SUBTRACT 1 from i
    END IF
END WHILE
END FUNCTION

Appendix B - Pseudo Code for the Algorithm to Calculate the Parameters c_{Bj} for j = 1,2,…n–1 for Beam Topology

FUNCTION calculating parameters cB()
INPUT: n parameters assigning lengths l_{Bj} to supports: n-element vector $\mathbf{b}$
OUTPUT: $(n-1)$-element vector $\boldsymbol{c}_B$
FOR i starts at 1, $i < n$, increment i DO ASSIGN to $\boldsymbol{c}_{Bi}$ the value zero
END FOR
FOR i starts at 1, $i \le n$, increment i DO
    IF b_i is not equal to 0 THEN ADD 1 to $\boldsymbol{c}_{B|b_i|}$
    END IF
END FOR
END FUNCTION

Appendix C - Pseudo Code for the Algorithms to Calculate the Coordinates of Supports and Hinges of Optimal Beam in a One-Dimensional Coordinate System with the Origin at the Left End of the Beam

FUNCTION calculating_support_coordinates()
INPUT: n parameters assigning lengths l_{Bj} to supports: n-element vector $\mathbf{b}$; the lengths l, l_E, l_H, l_{Bj} for $j = 1,2,..., n - 1$
OUTPUT: n coordinates of supports: n-element vector $\mathbf{s}$
IF b_1 is equal to 0 THEN ASSIGN to s_1 the value zero
ELSE ASSIGN to s_1 the value l_E
END IF
FOR i starts at 2, $i \le n$, increment i DO
    IF b_{i-1} is equal to 0 THEN
        IF b_i is equal to 0 THEN ASSIGN to s_i the value $s_{i-1} + l$
        ELSE IF b_i is greater than 0 THEN ASSIGN to s_i the value $s_{i-1} + l + l_H$
        ELSE ASSIGN s_i the value $s_{i-1} + l + l_{B|b_i|}$
    ELSE IF b_{i-1} is greater than 0 THEN
        IF b_i is equal to 0 THEN ASSIGN to s_i the value $s_{i-1} + l_{Bb_{i-1}} + l$
        ELSE IF b_i is greater than 0 THEN ASSIGN to s_i the value $s_{i-1} + l_{Bb_{i-1}} + l_H$
        ELSE ASSIGN s_i the value $s_{i-1} + l_{Bb_{i-1}} + l + l_{B|b_i|}$
    ELSE
        IF b_i is equal to 0 THEN ASSIGN to s_i the value $s_{i-1} + l_H + l$
        ELSE IF b_i is greater than 0 THEN ASSIGN to s_i the value $s_{i-1} + l + 2l_H$
        ELSE ASSIGN s_i the value $s_{i-1} + l_H + l + l_{B|b_i|}$
    END IF
END FOR
END FUNCTION
FUNCTION calculating_hinge_coordinates()
INPUT: the topological code of beam: n-element vector $\mathbf{t}$; n coordinates of supports: n-element vector $\mathbf{s}$; the length l_H
OUTPUT: $n - 2$ coordinates of hinges: $(n-2)$-element vector $\mathbf{h}$
FOR i starts at 1, $i < n - 1$, increment i DO

```
    IF $t_{n+1}$ is equal to 0 THEN ASSIGN to $h_n$ the value $s_{n+1}$
    ELSE IF $t_{n+1}$ is equal to 2 THEN ASSIGN to $h_n$ the value $s_{n+1} - l_H$
    ELSE ASSIGN to $h_n$ the value $s_{n+1} + l_H$
    END IF
END FOR
END FUNCTION
```

Appendix D - Pseudo Code for the Algorithm to Count the Number of Topological Classes

```
FUNCTION counting_number_of_classes()
INPUT: the number of supports $n$
OUTPUT: the number of classes $p^n$
ASSIGN to $p^n$ the value 0
CREATE an empty vector $\mathbf{c}_B$
FOR $c_E$ starts at 0 , $c_E \leq 2$, increment $c_E$ DO
    FOR $c_H$ starts at 0 , $c_H \leq n-2$, increment $c_H$ DO
      FOR $c_{B1}$ starts at $c_H$, $c_{B1} \leq c_E + c_H$, increment $c_{B1}$ DO
        IF ($c_E$ is equal to 2 ) and ($c_H$ is equal to $n-2$) and ($c_{B1}$ is odd) THEN
          SKIP to the next iteration of the loop
        END IF
        CALL generating_parameters_cB($c_E + c_H$, $c_{B1}$)
      END FOR
    END FOR
END FOR
END FUNCTION
FUNCTION generating_parameters_cB(expected_sum,current_sum)
(* Function generates parameters $c_{Bj}$ for $j = 2,... n-1$ *)
IF vector $\mathbf{c}_B$ has $n-1$ elements THEN
    IF expected_sum is equal to current_sum THEN ADD 1 to $p^n$
    END IF
    RETURN
END IF
IF expected_sum is less than current_sum THEN RETURN
END IF
FOR $i$ starts at last element of vector $\boldsymbol{c}_B$, $i \geq 0$, decrease $i$ DO
    ADD $i$ at the end of vector $\boldsymbol{c}_B$
    ADD $i$ to current_sum
    CALL generating_parameters cB(expected_sum,current_sum)
    DELETE last element of vector $\boldsymbol{c}_B$
    SUBTRACT $i$ from current_sum
END FOR
END FUNCTION
```