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Theory of Weightless Sagging Elasto-Flexible Cables

Abstract

In this Paper, a theory of single sagging planer weightless elasto-flexible cables is proposed. In reference to the instantaneous natural state, hypoelastic rate- type constitutive equations and third-order nonlinear differential equations of planer motion for massless cables with nodal masses are derived. New concepts like configurational complementary potential and contra-gradient principles are proposed. Typical dynamic response of these elastic cables with different sustained loads, axial rigidity and sag/span ratios is determined. Modal, subharmonic and internal resonances, jump and beat phenomena, etc., are predicted. A new mode of damping of 'free' vibration of the system perturbed from its equilibrium state is identified. Significance of the proposed theory of weightless sagging elasto-flexible cables is discussed.

Keywords

Weightless cables, configurational nonlinearity, rate- type hypoelastic constitutive equations, third order equations of motion, new damping mode

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1 INTRODUCTION

Most of the theoretical investigations on cable mechanics deal with nonlinear elastic single heavy sagging cables with or without concentrated loads (Irvine and Sinclair 1976; Nayfeh and Pai 2004; Impollonia et al. 2011; Lacarbonara 2013; Coarita and Flores 2015). The discrete finite elements are demonstrated to be better than the continuous catenary elements (Abad et al. 2013). A spatial two-node catenary cable element has been developed by Thai and Kim for nonlinear static and dynamic analysis of cables under self-weight and concentrated loads (Thai and Kim 2011). A maximum complementary energy principle has been formulated for stress-unilateral and physically nonlinear cables (Santos and Paulo 2011).The primary focus of these research investigations has been to devise efficient analytical and computational methods to predict the static and dynamic response of such nonlinear cable structures. There exists a multiplicity of approximate theories proposed for limited ranges of structural parameters valid for different applications. It has been established that, in the limiting case of shallow cables, the theory of sagging inextensible cables does not reduce to the theory of stretched strings (Irvine and Caughey 1974; Benedettini et al. 1986; Tsui 1990).

The linear modal frequencies of nonlinear cables are shown to be determined by system parameters based upon their geometric and elastic properties (Irvine and Caughey 1974; Triantafyllou and Grinfogel 1986). In this parameter space, nonlinear geometric, elastostatic and elastodynamic vibration modes are identified (Lacarbonara et al. 2007). Primary and subharmonic resonances as well as multiple internal resonances have been investigated by many researchers. Condition for stability of dynamic response near the simultaneous primary internal resonances is established (Kamel and Hamed 2010).

The popular linear approach is premised upon the linearized equations of motion. The initial tangent stiffness matrix used all through the analysis comprises of the elastic and geometric stiffness matrices. Such linear approach is not equipped to simulate the various characteristic dynamical phenomena of the essentially nonlinear cable structures (Volokh et al. 2003). Indeed, the conventional stability functions for incorporating the effect of axial force on lateral stiffness of elastic beam-columns are still recommended for cable analysis (Thai and Kim 2008). Gap Functional is derived as a new term in the expression for the potential energy for the elastic cables (Santos and Paulo 2011).

Flexible cables are known to lack unique natural state as well as stiffness in their passive state (Benipal 1992; Kim and Lee 2001; Koh and Rong 2004). Generally, their equilibrium state under the self- weight is assumed to play the role of natural reference state for describing their response to the additionally applied loads (Nayfeh and Pai 2004; Lacarbonara 2013). Here, such an approach according special status to self- weight is not followed. Instead, the weightless cable is considered to be subjected to only the concentrated loads.In such a case, the intermodal cable segments are straight lines in contrast to the catenary form assumed by them under distributed self- weight. Because of this fact, the theory of such weightless cables is often considered to constitute a special, rather simpler and unattractive, case of the theory of heavy cables under concentrated loads (Santos and Paulo 2011). Such indeed is not the case as demonstrated by the present Paper.

Theories of structures lacking unique natural state are not available. However, in continuum mechanics, the theory of hypoelasticity was proposed by Truesdell in 1955 for materials lacking unique natural state. Constitutive equations for such materials are of rate-type which does not demand the specification of a unique reference state (Truesdell and Noll 1965; Maugin 2013). Motivated by the theory of hypoelasticity, quasi-static incremental constitutive equations are developed by Benipal for a weightless single sagging cable under two vertical concentrated loads. Configuration-defining nodal coordinates and small elastic displacements are shown to respectively be zero and first order homogeneous functions of the applied loads. Configurational nonlinearity associated with dependence of the natural state upon loads is distinguished from the geometrical nonlinearity associated with finite elastic displacements as well as from the material nonlinearity associated with nonlinear stress-strain curve in tension (Benipal 1992). The same approach is adopted in this Paper.

Third order equations of motion, derived later in this Paper, have also been used to describe the dynamic response of many mechanical systems like elastically coupled isolator system, relaxation type of vehicle suspension, nonlinear control systems, etc. (Srinivasan 1995). Jerk or triceleration as the time rate of change of acceleration appears in the third order jerk equations of motion (Gottlieb 2004).

Also, Haan and Sluimer proposed a third order linear differential equation of motion for SDOF concrete, steel and bituminous materials to simulate the observed three modes of damping of their free vibrations (Haan and Sluimer 2001). Apart from the oscillatory and non-oscillatory free vibration modes of conventional underdamped and overdamped structures respectively, the third damping mode involving damped oscillatory response with the vibration amplitudes of the same sense in some initial cycles predicted by these researchers is predicted here for cables as well. Conditions for the oscillatory and nonoscillatory solutions of conventional third order differential equations are well understood (Padhi and Pati 2014).

The objective of the present Paper is to investigate the static and dynamic behaviour of weightless elasto-flexible cables. New rate-type constitutive equations and third order nonlinear differential equations of motion are derived for such cables under concentrated loads and with lumped masses. Using this proposed theory, free and forced vibration response of two node single cables is predicted. The scope of the Paper is limited to weightless sagging planer elasto-flexible cables which are linear elastic in axial tension. The proposed theoretical formulation and the predicted dynamic response are critically evaluatedand fruitful areas of further research are identified.

2 CONSTITUTIVE EQUATIONS

Consider a weightless planer inextensible flexible cable fixed at one end at A and with n nodes and 2n degree of freedom as shown in Fig. 1(a). Let (x_{2r-1}, x_{2r}) and (P_{2r-1}, P_{2r}) represent respectively the equilibrium state nodal coordinates and the nodal forces acting at the rth node. Here, S_r and T_r represent respectively the length the cable segment ending at node r and tensile forces in it. The following n geometrical relations apply:

$$(x_{2r-1} - x_{2r-3})^2 + (x_{2r} - x_{2r-2})^2 = S_r^2 r = 1, 2, 3, \ldots\ldots, n \tag{1}$$

The equilibrium equations for the r^{th} node are stated as

$$P_{2r-1} = T_r \frac{x_{2r-1} - x_{2r-3}}{S_r} - T_{r+1} \frac{x_{2r+1} - x_{2r-1}}{S_{r+1}} P_{2r} = T_r \frac{x_{2r} - x_{2r-2}}{S_r} - T_{r+1} \frac{x_{2r+2} - x_{2r}}{S_{r+1}} \tag{2}$$

For the n^{th} node, one obtains

$$P_{2n-1} = T_n \frac{x_{2n-1} - x_{2n-3}}{S_n} P_{2n} = T_n \frac{x_{2n} - x_{2n-2}}{S_n} \tag{3}$$

Using the above two equations, two expressions for T_n in terms of nodal forces are obtained. Similarly, the expressions for T_{n-1},T_{n-2},.....,T_1 can be developed. Equating these expressions for each of these segment tensile forces, one obtains another set of n equations stated in terms of unknown nodal coordinates x_i and nodal forcesP_i. The unknown 2n nodal coordinates can be determined by solving these 2n simultaneous equations. Because of the nonlinearity of n geometrical relations, explicit expressions for nodal coordinates cannot be derived in terms of the known nodal loads. However, for numerically prescribed nodal loads, these nodal coordinates can be determined using Newton-Raphson Method. Then, the tensile forces in the cable segments can also be determined.

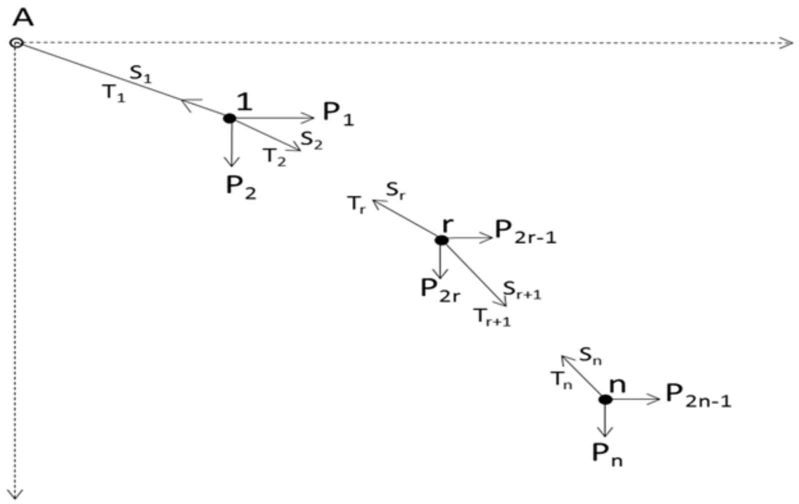

Fig. 1(a): Cable with n nodes fixed at one end.

Closer scrutiny of the simultaneous equations reveals the dependence of the nodal coordinates (x_i)on the relative magnitudes of the nodal forces. To be exact, the configuration-defining nodal coordinates (x_i) turn out to functions homogeneous of order zero of the nodal loads. Similarly, the tensile forces (T_r) in the segments of the cable undergoing small elastic displacements are first order homogeneous functions of the applied loads. Under proportional monotonic loading, the relative magnitudes of theapplied loads do not change. Consequently, under such load variations, the configuration-defining nodal coordinates (x_i)do not change for inextensible as well as elastic cables. Of course, elastic cables do undergo elastic displacements $(u_i)$and their deformed state nodal coordinates (y_i)do change under such loading.However, for the case of small elastic displacements, elastic displacements (u_i)as well as the segment tensile forces (T_i)increase linearly with proportional monotonic loading.

In view of this, configurational complementary energy (Ω_c) of the cable is defined here for the first time as a function homogeneous of order unity of the applied nodal loads. The following expressions are proposed for it:

$$\Omega_c = P_i x_i \qquad i = 1, 2, 3,...., 2n \qquad \text{and} \qquad \Omega_c = T_r S_r \qquad r = 1, 2, 3,..., n \tag{4}$$

The first expression for Ω_c can be interpreted as the work done by the applied loads during loading, while the second expression is interpreted as the energy stored in the inextensible cable in its equilibrium state. Thus, the expression for complementary potential energy for inextensible cables can be stated as

$$\Pi_c = T_r\, S_r - P_i x_i \tag{5}$$

The tangent configurational compliance coefficients $\left(D_{ij} = \partial x_i / \partial P_j\right)$ are obtained as functions homogeneous of order (-1) of the applied nodal forces. Using Euler's equations for homogeneous functions, the following valid relations can be easily derived:

$$D_{ij} P_j = 0 D_{ij}\, P_i\, P_j = 0 i,\ j = 1,2, \quad ....2n \tag{6}$$

The symmetric matrix D_{ij} can be observed to singular. Using this equation, a Castigliano- type theorem is stated as $\partial\Omega_c/\partial P_i = x_i$. It is worth noting that an elastic structure resists the applied loads in its deformed equilibrium state configuration. Assuming small elastic displacements, the equilibrium configuration of the corresponding elastically inextensible cable is assumed to be the reference state for carrying out static analysis and for defining elastic displacements. Using Castigliano's Theorem, the nodal elastic displacements (u_i) of the cable assumed to be linear elastic in tension are obtained from the elastic complementary energy$(\Omega_e = (1/2)T_r\Delta_r\,;\;\; \Delta_r = T_rS_r/AE)$ as

$$u_i = \frac{\partial\Omega_e}{\partial P_i} = \left(\frac{1}{AE}\right) T_rS_r\, B_{ri}B_{ri} = \frac{\partial T_r}{\partial P_i} \tag{7}$$

The tangent elastic flexibility matrix coefficients $(f_{ij})$are then determined as below:

$$f_{ij} = \frac{\partial u_i}{\partial P_j} = \left(\frac{1}{A\,E}\right)\sum_{r=1}^{n}\left[\, S_rT_r\frac{\partial B_{ri}}{\partial P_j} + \; S_rB_{ri}\, B_{rj}\right] \tag{8}$$

Here, the applied loads are independent variables which determine the nodal coordinates(x_i), tangent configurational flexibility matrix D_{ij}, tensile forces T_r, configurational and elastic complementary energy potentials, Ω_c and Ω_e, small elastic displacements u_i and tangent elastic flexibility matrix coefficients f_{ij}. These dependent variables turn out to be homogeneous functions of the applied loads. Specifically, x_i and f_{ij} are zero order, u_i, T_r and Ω_c are first order, Ω_e is the second order and D_{ij} are (-1) order homogeneous functions of the nodal loads. Using Euler's Theorems for homogeneous functions, one obtains the following additional useful relations:

$$f_{ij}P_j = u_i 2\Omega_e = f_{ij}P_i\;P_j = P_iu_i \tag{9}$$

Based upon the above arguments, the following principles of contra-gradience can also be stated for elasto- flexible cables undergoing small elastic displacements:

$$B_{ij}P_j = T_iB_{ij}^T\Delta_i = u_jB_{ij}^TS_i = x_jB_{ij}^T(S_i + \Delta_i) = x_j + u_j = y_j \tag{10}$$

An elasto- configurational complementary potential, Ω, which can be defined as $\Omega = \Omega_c + \Omega_e$, is a mixed order homogeneous function of the applied loads. Thus,

$$y_i = \frac{\partial\Omega}{\partial P_i}N_{ij} = \frac{\partial^2\Omega}{\partial P_i\,\partial P_j}N_{ij}P_j = \left(D_{ij} + f_{ij}\right)P_j = u_i \tag{11}$$

Here, x_i denote the nodal coordinates of the undeformed elastic cable in its natural state attained by proportional unloading. This natural state configuration is identical with the corresponding inextensible cable in equilibrium with the same load set. This natural state plays the role of reference state for the elastic displacements. The symbols y_i denote the corresponding coordinates of the elastically deformed cable in its equilibrium state. It should be appreciated that unloading following some nonproportional load path leaves the cable in a configuration which is determined by the relative magnitudes of the loads just before completion of unloading process. This is in confirmation with the fact that flexible cables do not possess unique natural state configuration.

The incremental constitutive equations for the weightless sagging elasto-flexible cables under nodal loads are stated as below:

$$dx_i = D_{ij}\, dP_j du_i = f_{ij}\, dP_j\, dy_i = N_{ij}\, dP_j \tag{12}$$

These constitutive equations can be recast in the rate form as follows:

$$\dot{x}_i = D_{ij}\dot{P}_j\dot{u}_i = f_{ij}\dot{P}_j\dot{y}_i = N_{ij}\dot{P}_j \tag{13}$$

$$K_{ij} = N_{ij}{}^{-1}\dot{P}_i = K_{ij}\dot{y}_j P_i = K_{ij}u_j \neq K_{ij}y_j \tag{14}$$

To recapitulate, both the configuration-defining nodal coordinates (x_i) and the nodal elastic displacements (u_i) depend upon the nodal forces. However, unlike the elastic displacements from the undeformed equilibrium state, there is no concept of configurational displacements. Thus, elastic displacements which vanish upon unloading, though not the configuration- defining nodal coordinates or placements, are kinematic variables. Still, both the configurational and elastic nodal velocities are kinematic variables which appear in the rate-type constitutive equations.

In the approximate analysis presented above, nodal equilibrium is investigated in reference to the undeformed configuration. Exact static analysis requires that the deformed cable segment lengths and so the elastically displaced nodal coordinates must be considered. Since the deformed configuration is not known a priori, an iterative method of static analysis seems to be the most appropriate. In that case, y_i, u_i, f_{ij} and Ω_e are not homogeneous functions of the applied loads. However x_i, D_{ij} and Ω_c still remain functions homogeneous respectively of order zero, minus one and one of the loads.

3 EQUATIONS OF MOTION

Consider that the above elasto-flexible cable is provided with lumped nodal masses M_i and is vibrating under the action of applied nodal forces F_i. These applied forces are distinct from the instantaneous internal resistive nodal forces P_i. Using Newton's second law of motion, the second order coupled differential equations of motion are stated for MDOF lumped-mass damped dynamical systems:

$$P_i = F_i(t) - C_{ij}\dot{y}_j - M_{ij}\ddot{y}_j \tag{15}$$

Because of the inequality $(P_i \neq K_{ij}y_j)$ in equation (14), this equation cannot be stated in the form of conventional second order differential equation of motion. The rate of change of internal resistive force is obtained as

$$\dot{P}_i = \dot{F}_i - M_{ij}\dddot{y}_j - C_{ij}\ddot{y}_j \tag{16}$$

This derivation is valid if and only if the damping matrix is constant. To be specific, the damping matrix coefficients are assumed not to depend upon system response and also to be time- invariant. Inserting the second equality $(\dot{P}_i = K_{ij}\dot{y}_j)$ from equation (14) in the above equation, one obtains the following equation:

$$M_{ij}\dddot{y}_j + C_{ij}\ddot{y}_j + K_{ij}\dot{y}_j = \dot{F}_i \tag{17}$$

The above equation represents the third order nonlinear differential equation of motion for the damped cable structures. Here, the constant damping matrix C_{ij} is determined from the mass matrix $M_{ij}$and the initial stiffness matrix K_{ij0}corresponding to the equilibrium state as

$$C_{ij} = a_0\, M_{ij} + a_1\, K_{ij0} \tag{18}$$

The values of the coefficients $a_0$and $a_1$required for determining the damping matrix are determined by assigning the chosen damping ratios for the two lowest modes.

The initial value problem involving the third order differential equation of motion (17) demands the specification of three initial conditions, viz., $y(o)$, $\dot{y}(0)$ and$\ddot{y}(0)$ Here, the equation (15) of dynamic equilibrium yields $\ddot{y}(0)$ for the conventionally specified initial values of $y(o)$ and $\dot{y}(0)$ as below:

$$\ddot{y}(0) = M^{-1}(F(0) - P(0) - C\,\dot{y}(0)) \tag{19}$$

Here, $F(0)$ represents the initial value of the applied nodal forces. Of course, the initial values of the resistive nodal forces $P(0)$corresponding to known $y(0)$ are used for obtaining the initial stiffness matrix to be employed for predicting the dynamic response.

Of course, both the equation of dynamic equilibrium (15) and the third order differential equation of motion (17) have to be satisfied at all instants. Solution of the differential equation of motion yields instantaneous nodal velocities, nodal accelerations and the rates of nodal acceleration. Knowing the instantaneous velocities and accelerations, the instantaneous internal resistive forces are determined using equation (15). Then, the instantaneous values of x_i, u_i and y_i are obtained. Similarly, the instantaneous stiffness is determined from the instantaneous values of resistive forces (P_i) as discussed above. Similarly, the third order equation of motion for the inextensible cables is obtained as

$$L_{ij}\dddot{x}_j + C_{ij}\ddot{x}_j + \delta_{ij}x_j = D_{ij}\dot{F}_j L_{ij} = D_{ij}M_{ji} \tag{20}$$

In case, the coordinates of the nth node are specified ($x_{2n-1} = L$, $x_{2n} = H$), the nodal forces, $P_{2n-1}$and P_{2n}, come out as reactions. Still, the expression for Ω_c, as per equation (4), remains valid in its present form. The same conclusion remains valid for equation (6). However, the subscripts i and j in all the equations of motion can assume the values: 1, 2,...., 2(n-1). As in the classical theory of vibrations, the equations of motion and of dynamic equilibrium apply only to kinematic degrees of freedom. It is the dynamic behaviour of such cables with two fixed ends which is predicted in the succeeding Section. Unlike equation (8) forf_{ij}, it is not possible to derive similar single expression for all the tangent configurational flexibility coefficients D_{ij}. Instead, separate expressions for these coefficients are presented below:

$$D_{ij} = \frac{1}{R}\begin{bmatrix} s\,m^2\,x_2^2 & -s\,m^2x_1x_2 & s^2mx_2(x_4-H) & s^2mx_2(L-x_3) \\ -s\,m^2x_1x_2 & s\,m^2x_1^2 & -s^2mx_1(x_4-H) & -s^2mx_1(L-x_3) \\ s^2mx_2(x_4-H) & -s^2mx_1(x_4-H) & s^3(x_4-H)^2 & s^3(L-x_3)(x_4-H) \\ s^2mx_2(L-x_3) & -s^2mx_1(L-x_3) & s^3(L-x_3)(x_4-H) & s^3(L-x_3)^2 \end{bmatrix} \tag{21}$$

$$R = m\,(A_1x_2 - A_2x_1) + \{A_3(x_4-H) + A_4(L-x_3)\}\,s$$

$$m = s + Hx_3 + L\,x_2 - H\,x_1 - L\,x_4, s = x_1x_4 - x_2x_3$$

$A_1 = r\,(Lx_4 - Hx_3)(x_2 - x_4) + P_2\, m\, s \qquad A_2 = r\,(Lx_4 - Hx_3)(x_3 - x_1) - P_1 m\, s$

$$A_3 = r(Lx_2 - Hx_1)(x_4 - x_2) + P_4\, s^2 A_4 = r(Lx_2 - Hx_1)(x_1 - x_3) - P_3 s^2$$

Linear modal frequencies for the elasto- flexible cables are determined from the initial stiffness matrix as $\sqrt{M^{-1}K}$. For ready reference, the sole linear modal frequency ω of the inextensible cable is derived as $\sqrt{R/s(m^2 M_1 S_1^2 + s^2 M_2 S_3^2)}$.

4 PREDICTED DYNAMIC RESPONSE

The numerical predictions presented in this Section pertain to the response of a particular sagging cable structure with the following details as shown in Fig. 1 (b):

$$L = 300\text{ m} \;\; H = 20\text{ m} S_1 = 130\text{ m} S_2 = 100\text{ m} S_3 = 110\text{ m}$$

$A = 0.005\text{ m}^2 E = 2 \times 10^{11}\text{ N/m2} \qquad M_1 = 8000\text{kg} \qquad M_2 = 14000\text{kg}$

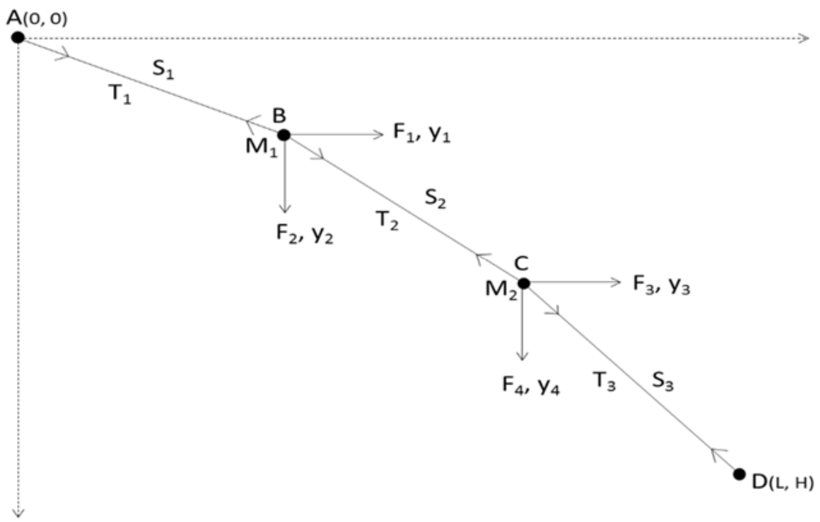

Fig. 1(b): Cable structure fixed at both ends with lumped masses.

The damping of the cable structures is known to be quite low: (Gimsing and Georgakis, 2012; Johnson, Baker, Spencer and Fujino, 2007; Chang, Park and Lee, 2008). Here, equal modal damping ratios (ξ = 0.01) are assigned to the two lowest modes. Using the third order nonlinear differential equations of motion (17), the dynamic response of cable structures is determined. Two numerical integration schemes, viz. Runge-Kutta Method and Newmark–Beta Method, are employed to cross-validate the predicted response. The forcing function is assumed to be of the form: $F(t) = F_0 + F_L \sin \omega t$. It is the variation of z_i with forcing frequency ω_f which is depicted in the frequency domain response (FDR) plots. Here, z_i representing the nodal elasto- configurational displacements from the equilibrium state configuration $y_i(0)$are given as $z_i(t) = y_i(t) - y_i(0)$, where, $y_i(0) = x_i(0) + u_i(0)$.

Also, instantaneous value of z_i are obtained by superposing the configuration displacements (z_{ic}) and elastic displacements (z_{ie}) from the equilibrium state as follows:

$$z_i(t) = z_{ic}(t) + z_{ie}(t) z_{ic}(t) = x_i(t) - x_i(0) z_{ie}(t) = u_i(t) - u_i(0) \tag{22}$$

As evidenced by the z_1 displacement waveform shown in Fig. 2, a particular highly damped cable ($\xi = 0.2$) perturbed by some chosen initial nodal velocities proportional to the third linear mode velocity vector ($\omega_{n3} = 29.0348$ rad/s) executes damped 'free' vibrations about the equilibrium state. In this damping mode, the oscillatory motion between the second and third peaks located at 0.279 s and 0.507 s occurs with amplitudes of the same sense. This mode of damping differs from the damping modes typical of conventional underdamped and overdamped structures. The initial high frequency response is observed to decay fast and is then followed by damped response at the lowest modal frequency. The vibration response predicted here should be interpreted in the light of the fact that the prescribed initial nodal velocities are equivalent to the cable structure being subjected to impact.

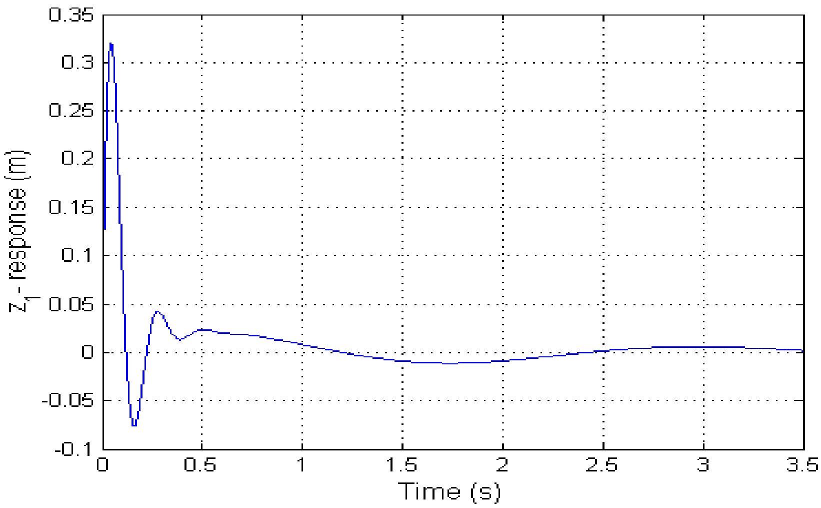

Fig. 2: z_2 - waveform for damped modal vibrations.

Typical partial FDR for the elastic cable under a particular sustained load set F_0: $[300, 1100, 450, 2000]$ kN and subjected to a small arbitrary sinusoidal forcing function F_L: $[-6, 22, -18, 120]$ kN is presented in Fig. 3(a). Both regular ($\omega = 5.10$ rad/s) and irregular sub-harmonic ($\omega = 1.27$ rad/s) resonance peaks are observed in addition to the fundamental peak ($\omega = 2.50$ rad/s). The nodal displacement $z_2$waveform and the corresponding phase plot shown in Fig. 3(b) present an initial unstable small-amplitude T2 regularsubharmonic resonance response, called jump phenomenon, at the forcing frequency of 5.10 rad/s. After an elapse of about 500s, the system gradually attains stable large-amplitude steady state fundamental response at the modal frequency of 2.50 rad/s.

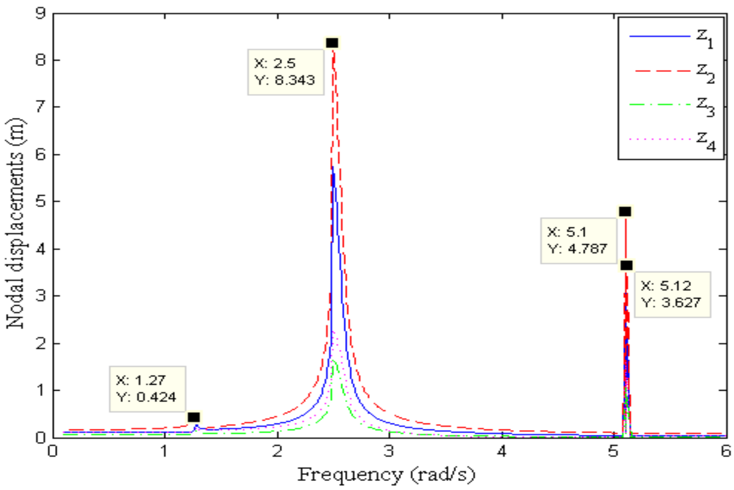

Fig. 3(a): Typical FDR for general forcing function.

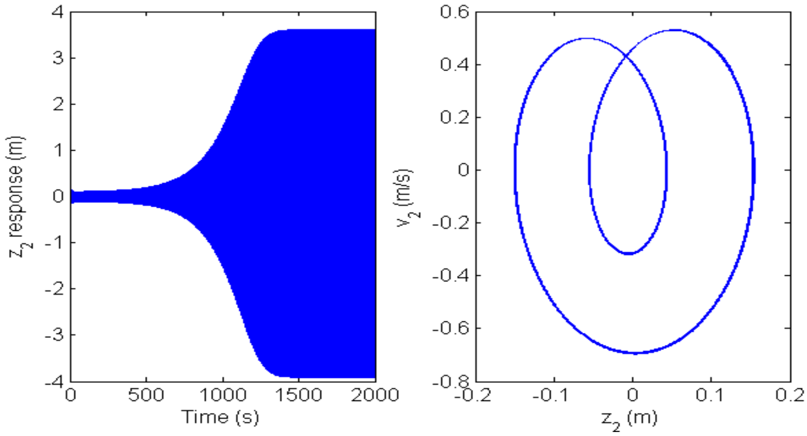

Fig. 3(b): z_2- waveform and phase plot ($\omega = 5.1$ rad/s).

The typical effect of nodal peak sinusoidal forces on the FDR is presented in Fig. 4 for a particular case of general sustained and harmonic loading: F_0: $[300, 1100, 450, 2000]$kN and F_L: $\gamma[-300, 1100, 900, 2000]$ kN. The nodal peak sinusoidal forces are increased by increasing the value of the parameter γ. Despite very large nodal displacements from the equilibrium state, the fundamental and the subharmonic resonance frequencies are not affected much with increase in sinusoidal forces, but the resonance peak z_2 amplitudes are predicted to increase in both the cases. Like linear structures, the fundamental peak amplitudes are predicted to increase more or less proportionately with the applied peak sinusoidal forces, while the regular subharmonic peak amplitudes register disproportionately higher increase.

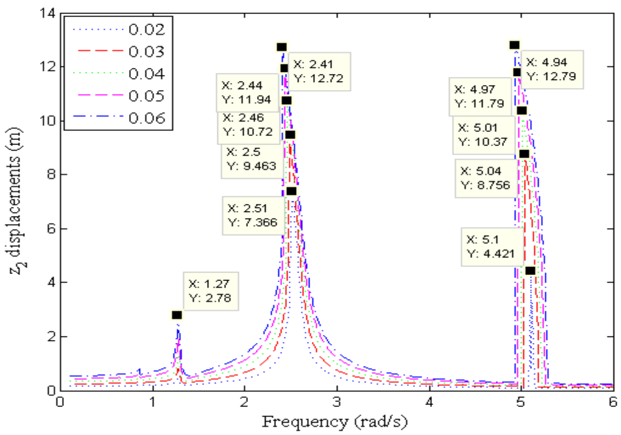

Fig. 4: Effect of peak sinusoidal forces on FDR.

Mechanical systems with the modal frequencies bearing an integral ratio with each other are known to exhibit the phenomenon of internal resonance at the appropriately adjacent modal forcing frequency [25]. The partial FDR plot for the one-to-one resonance with peaks at forcing frequencies $(12.05\ \mathrm{rad/s},\ 12.75\ \mathrm{rad/s})$ is shown in Fig. 5(a) for the proportional peak sinusoidal forces. Such a crossover at the two equal linear natural frequencies $(\omega_{n1} = 12.3591\ \mathrm{rad/s},\ \omega_{n2} = 12.9097\ \mathrm{rad/s})$ occurs at a particular sustained loads $F_0: 464.764288\,[30, 110, 45, 200]\ \mathrm{kN}$ and peak sinusoidal loads $F_L: 0.02\ F_0$. The z_4 peak amplitude exceeds the z_2 peak amplitudes at the lower forcing frequency, while these vertical peak amplitudes are approximately equal at the higher forcing frequency. The $z_4$waveform for the forced vibration for one-to-one resonance case at low damping ($\omega = 12.63$ rad/s, $\xi = 0.002$) shown in Fig. 5(b) exhibits typical beat phenomenon. The oscillatory motion occurs at the response frequency of 12.56 rad/s with the beat frequency of about 0.37 rad/s. It can be observed that the beat frequency approximately equals half the difference of the above linear modal peak frequencies. As expected, the beat phenomenon being essentially transient, the steady state forced vibration response is restored after some time.

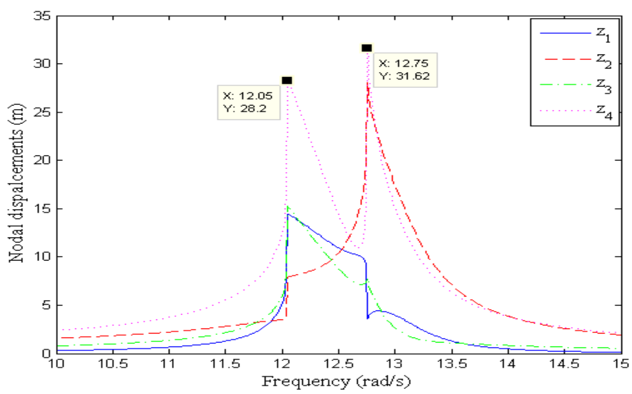

Fig. 5(a): FDR for one-to-one internal resonance with proportional harmonic loading.

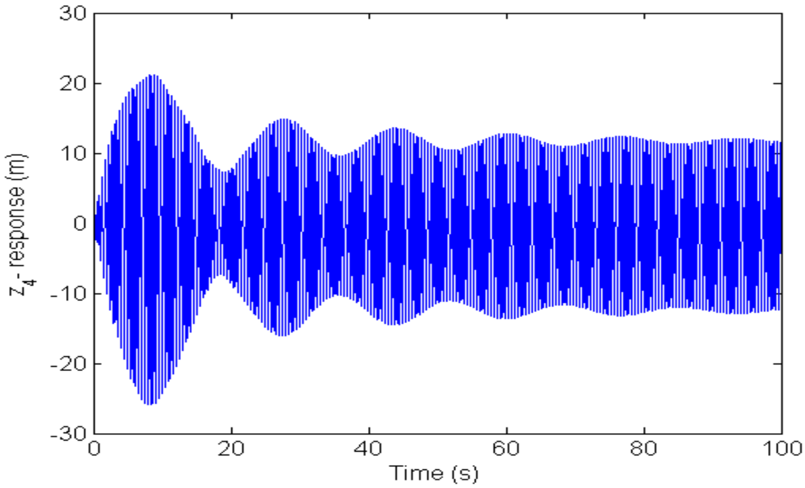

Fig. 5(b): Typical beat phenomenon.

The predicted dynamic response for the cables with low sag/ span ratio is presented in the form of approximate and exact FDR plots respectively shown in Fig. 6 and Fig. 7(a) for a particular case ($S_1 = 115.33\text{m},\ S_2 = 88.72\text{m},\ S_3 = 97.59\text{m}$). Approximate analysis can be observed to predict lower modal frequencies but much higher peak amplitudes for the first two vibration modes than the exact analysis. In both the vibration modes, the horizontal nodal displacements are much lesser than the vertical nodal displacements. However, vibration amplitude z_4 dominates in the first mode while z_2 dominates in the second mode.

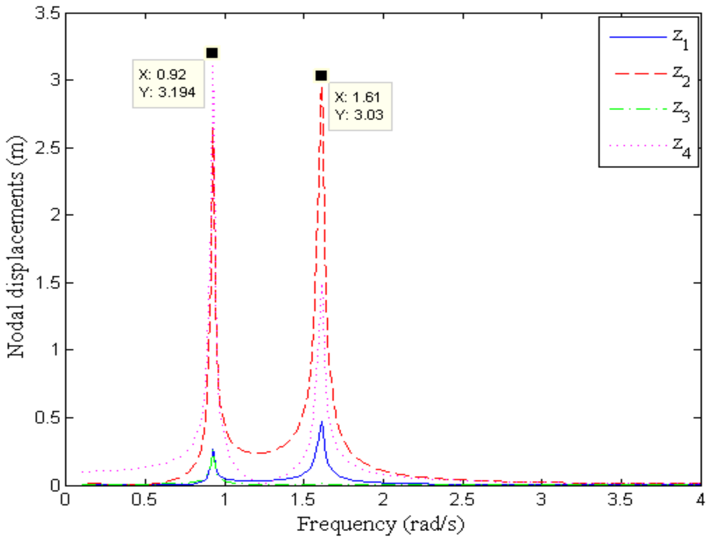

Fig. 6: Approximate FDR for cables with low sag/span ratio.

Focusing only upon the dominant vertical nodal displacements, the Lissajous plots shown in Fig. 7(b) reveals the first exact mode ($\omega = 1.84$ rad/s) to be symmetric wherein both z_2 and z_4 are of the same sense. In contrast, second mode ($\omega = 2.56$ rad/s) is anti-symmetric with the opposite sense of z_2 and z_4 amplitudes. The peak amplitudes in these two modes are of approximately the same order. However, the change in the tensile force (T_1) in the same segment for the second mode is observed to be many times more than that for the first mode. Fig. 7(c) shows the component configurational and elastic waveforms for these two vibration modes. It can be observed that the symmetric mode response is essentially elastic. In contrast, the elastic and configurational contributions to the second anti-symmetric mode amplitudes are of the same order. It can be observed that, for the second anti-symmetric mode, the positive and negative excursions of both the configurational and elastic components from the equilibrium state are different. Perhaps, this is why the tensile force T_1 is observed in Fig. 7(b) to exhibit much higher positive deviations than the negative deviations from its value in the equilibrium state.

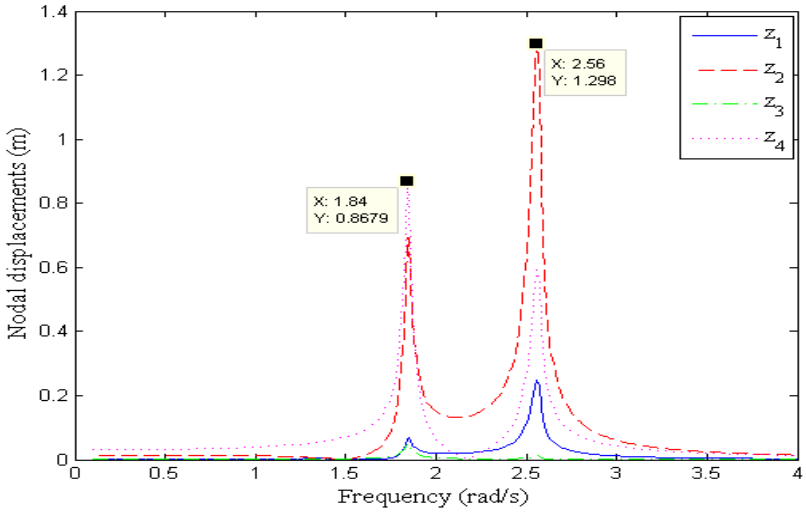

Fig. 7(a): Exact FDR for cables with low sag/span ratio.

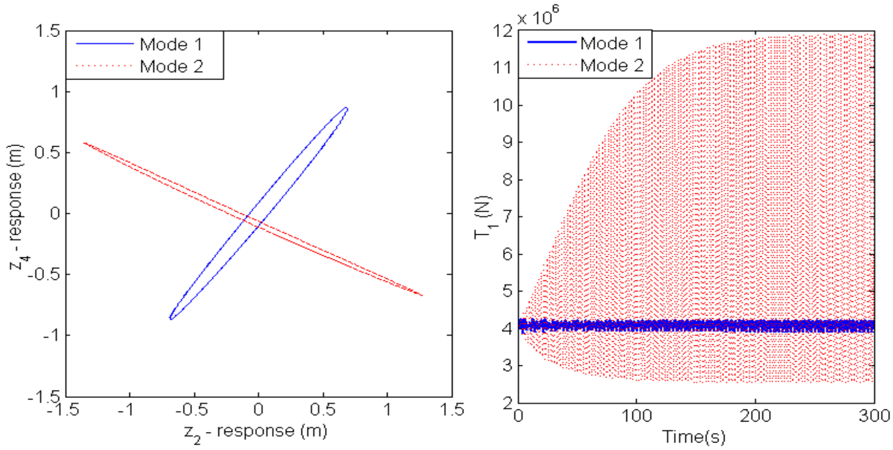

Fig. 7(b): Modal Lissajous plots and modal tension T_1 waveforms.

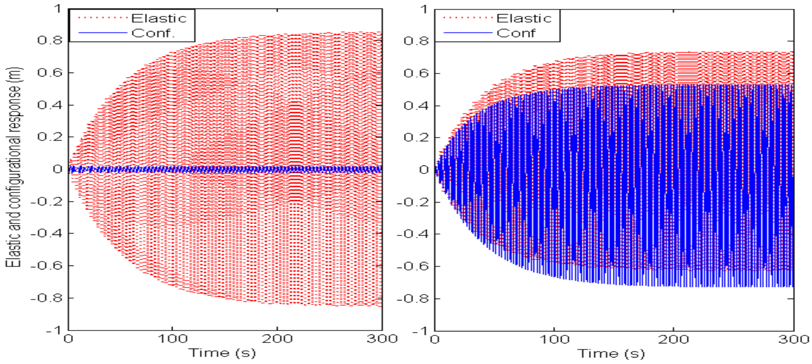

Fig. 7(c): z_4 - waveforms for mode 1($\omega_f = 1.84$ rad/s) and mode 2 ($\omega_f = 2.56$rad/s).

In this Paper, the dynamic response of inextensible cables is determined as the limiting case of elastic cables with very high axial elastic rigidity. The modal frequencies of the particular 'inextensible' cable ($EA = 10^{12}N$) investigated using exact analysis are 0.8154, 401.0171, 864.3224, and 1472.6295 rad/s. The lowest frequency is interpreted as the only modal frequency (ω_{inext}) of the inextensible cable whose other modal frequencies are known to be infinitely high.

Frequency domain response curves of the inextensible cable in its equilibrium state and subjected to a particular peak sinusoidal load sets are shown in Fig. 8(a). The inextensible cable exhibits both regular and irregular subharmonic resonance in addition to the fundamental resonance. The inextensible cables can experience only such antisymmetric configurational nodal vibrations. Waveforms shown in Fig. 8(b) depicting the fundamental elastic and configurational amplitudes (z_{2e} and z_{2c}) for the second degree of freedom confirms this fact. The regular subharmonic peak is split into two peaks at 1.62 and 1.64 rad/s respectively. Displacement waveforms for the first case ($\omega = 1.62$ rad/s) depicted in the Fig. 8(c) shows rapid increase to higher amplitude steady state vibration after an elapse of certain period of about 1000s. The corresponding FFT plot also reveals the two response frequencies (0.8325 rad/s and 1.649 rad/s) respectively equal approximately to the fundamental and the regular subharmonic resonance frequencies.

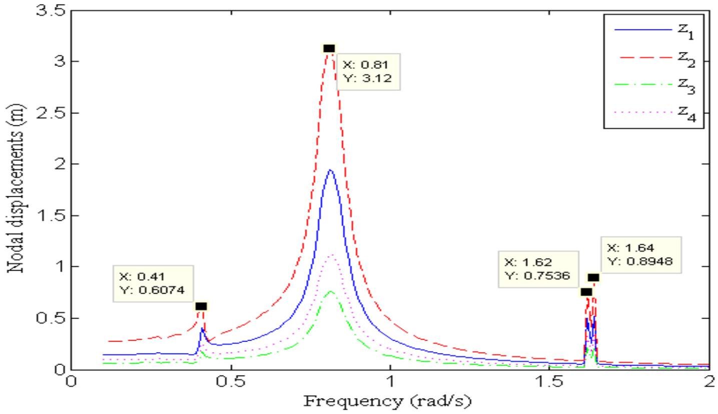

Fig. 8(a): Exact FDR for inextensible cables under general harmonic loading.

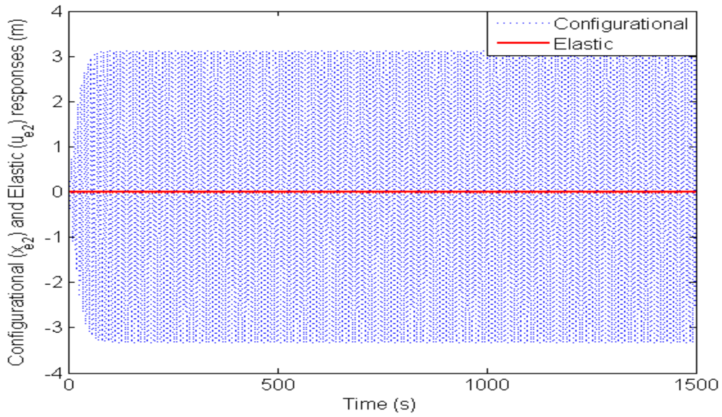

Fig. 8(b): Configurational and elastic response plot (ω=0.81 rad/s).

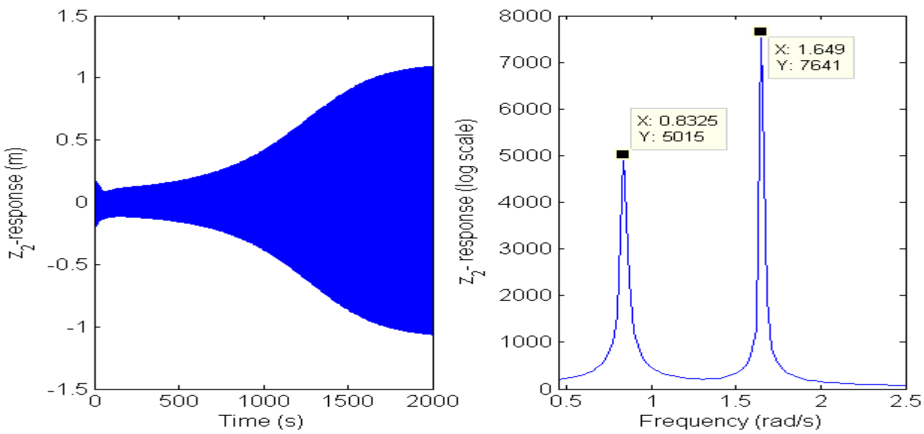

Fig. 8 (c): z_2 - waveform and FFT plot (ω=1.62 rad/s).

5 DISCUSSION

Elasto- flexible slack cables are known to lack unique natural passive state. This is confirmed here by the singularity or non- invertibility of their tangent configurational flexibility matrix. Of course, the load- dependent equilibrium states of the cables are unique. When undergoing small elastic displacements, such cables are shown here to belong to the class of homogeneous mechanical systems. Of course, there is no limit on the total nodal displacements. Using Euler's Theorems for homogeneous functions, many new relations delineating the characteristic structural response of these cable structures are established. Cables exhibit configurational response only under non-proportional load variations. This is why configurational response could not be identified as distinct from the elastic response in earlier investigations dealing with cables under single parameter self- weight. Configurational nonlinearity is always present in cable response. The approximate and exact methods of cable analysis used in this Paper respectively belong to the class of geometrically linear and nonlinear approaches.

Here, the complementary configurational and elastic potentials are shown to be functions homogeneous of order one and two respectively of the loads for the cables undergoing small elastic displacements. The configurational complementary potential proposed here for the first time for the inextensible cable resembles Santos-Paulo's Gap Functional.Also, new contra-gradient principles for inextensible cables as well as cables undergoing small elastic displacements are proposed here. These principles are distinct from the well- known contra-gradient principles for linear elastic structures as well as those used in the incremental form by other researchers for nonlinear elastic cable structures (Thai and Kim, 2008). Here, the tangent transformation matrix B is used for relating nodal loads, displacements, and coordinates with member forces, displacements and lengths.

In this Paper, rate- type constitutive equations relating nodal velocities with the loading rates are derived. The tangent elasto- configurational flexibility matrix N is determined as the sum of tangent configurational and elastic flexibility matrices Dand f. This approach is in contrast to the conventional procedure of obtaining the elasto-geometrical stiffness matrix (k_{eg}) by adding the tangent elastic and geometric stiffness matrices k_e and k_g. The approach followed here to establish the tangent stiffness matrix is different even from other approaches proposed for the nonlinear cable structures (Volokh et al. 2003; Nayfeh and Pai 2004; Lacarbonara 2013). Others (Chang et al. 2008; Abad et al. 2013) obtain the tangent stiffness matrix by inverting the tangent flexibility matrix which resembles the elasto- configurational flexibility matrix proposed here. Direct formulation of the geometric stiffness matrix popular in conventional elastic structures undergoing finite deformations and incorporating the effect of loads on their stiffness is not valid here. The stiffness of the cables is affected by the loads even when the elastic displacements are infinitesimally small or are altogether absent as in the case of inextensible cables.

Purely configurational response is predicted for the inextensible cables (N= D). Also, the cables under very high sustained loads (approximatelyN = f) and small variable loads are predicted to display dominantly elastic response. However, in the latter case, the elastic displacements become unacceptably high. Reduction of elastic displacements by increasing axial rigidity of the cable results in comparable configurational displacements. Such persistence of configurational response in all realistic situations renders invalid the popular purely elastic methods of analysis of cable structures under high sustained loads.

Using the rate- type constitutive equations, the third order nonlinear differential equations of motion are explicitly proposed for the MDOF elasto-flexible cables. Equivalently, the proposed equations of motion (17) can be recast in the incremental form used earlier ($M\Delta\ddot{y} + C\,\Delta\dot{y} + K\,\Delta y = \Delta F$)by the other researchers (Thai and Kim 2011). In both the cases, K represents the tangent stiffness matrix and the tangent viscous damping matrix is obtained as linear combination of mass and tangent stiffness matrices. However, the initial value problem formulated in the Paper in connection with cable dynamics essentially differs from the conventional initial value problems involving second order differential equations of motion.Even the governing equations of motion proposed here differ from the existing third order differential equations of motion as presented in the Introduction. In some proposed formulations (Volokh et al. 2003; Chang et al. 2008; Thai and Kim 2011; Abad et al. 2013), the tangent stiffness matrices are evaluated merely for the pragmatic purpose of stating the equations of motion in the chosen incremental form. In contrast, the rate- type constitutive equations, the tangent configurational and elastic flexibility matrices, and the third order differential equations of motion appear in the present theory as the only appropriate approach for these mechanical systems lacking unique natural state.

The main point of departure of this Paper is the third order differential equations of motion for these hypoelastic MDOF cable structures. The proposed theory is illustrated for the two node four DOF cable, the simplest structures capable of exhibiting the characteristic configurational response. No attempt has been made here to empirically validate the theoretical predictions. Of course, there is an almost total absence of experimental and analytical studies dealing with weightless cables. Still, some qualitative compatibility with some other investigations is noted below:

The theory of nonlinear vibrations of cables proposed in this Paper encompasses complete ranges of sustained loads, axial elastic stiffness values and sag/span ratios. A particular harmonically excited sagging cable has been predicted in this Paper to exhibit the transient beat phenomenon. Such beat phenomenon has earlier been predicted for guy cables for telecommunication masts (Oskoei and McClure 2011) and for sagging heavy elastic cables (Benedettini and Rega 1987). The jump phenomenon associated with the instability of the initial transient response after some finite duration predicted here is also observed by other investigators (Srinil and Rega 2008). Peak nodal displacement amplitudes in the vibrating conventional elastic structures have design significance as their displacements determine the internal forces caused by dynamic loading. In contrast, as demonstrated here, larger total nodal displacements but with smaller elastic component from the equilibrium state may not imply higher tensile forces.

The scope of the present Paper is restricted to planer behaviour of weightless sagging cables. Despite, perhaps because of, the restricted chosen scope of the Paper, new results not available in the existing literature on sagging cables (Tsui 1990; Starossek 1994; Rega 2004; Nayfeh and Pai 2004; Lacarbonara 2013) have been deduced. The proposed theory can obviously be extended to their spatial behaviour. An attempt can be made to simulate the distributed self-weight of the cable by many concentrated loads. Also, new formulations are required for handling the slackness of cable segments during motion. More powerful techniques of nonlinear dynamical systems theory can be used for exploring the behaviour of these structures obeying characteristic third order coupled nonlinear differential equations of motion.

6 CONCLUSIONS

In this Paper, a new theory of the single sagging weightless elasto-flexible planer cables carrying nodal lumped masses and concentrated loads is proposed. Recognizing the lack of uniqueness of their natural state, hypoelastic rate-type constitutive equations and third order nonlinear differential equations of motion are derived. To the best of Authors' knowledge, such constitutive equations and equations of motion have not been proposed earlier. Cables undergoing small elastic displacements are proved to belong to the class of homogeneous mechanical systems. Configurational nonlinearity exhibited by the cables is shown to be distinct from the conventional physical and geometrical nonlinearities. Like elastic complementary energy potential, a new configurational complementary energy potential is identified for inextensible as well as elastic cables. New contra-gradient principles are also proposed. Sustained loads, axial elastic stiffness and sag/span ratio are observed to affect their structural response. In addition to the fundamental modal resonances, cables are shown to exhibit the regular and irregular subharmonic resonances. Beat and jump phenomena as well as the third damping mode have been predicted. The theory of single sagging planer cables proposed in this Paper is critically evaluated and some fruitful areas of further research are identified.

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Notations

A – cross-sectional area of the cable

a_0, a_1– damping coefficients

B_{ij} –tangent transformation matrix; i=1, 2, 3; j = 1, 2, 3, 4

C_{ij}– damping matrix; i ,j = 1, 2, 3, 4

D_{ij}– tangent configurational flexibility matrix; i ,j = 1, 2, 3, 4

E – modulus of elasticity of the cable material

F_{ij}, $\dot{F}_{ij}$– applied nodal load and load-rate vectors; i = 1, 2, 3, 4

F_0– applied constant sustained load

F_L– applied peak harmonic load

f_{ij}– tangent elastic flexibility matrix; i, j = 1, 2, 3, 4

g – acceleration due to gravity

K_{ij}– tangentelasto-configurational stiffness matrix; i, j = 1, 2, 3, 4

k_e– elastic stiffness matrix

k_g– geometrical stiffness matrix

k_{eg}–elasto-geometrical stiffness matrix

L, H – horizontal and vertical distance between fixed supports

M_{ij}– mass matrix

N_{ij}– tangentelasto-configurational flexibility matrix; i, j = 1, 2, 3, 4

P_i, $\dot{P}_i$ –resistive nodal load and load-rate vectors; i = 1, 2, 3, 4

S_r– length of cable segments; r = 1, 2, 3

T_r– tension in segments; r = 1, 2, 3

u_i, $\dot{u}_i$– nodal elastic displacement and velocity; i = 1, 2, 3, 4

x_i, $\dot{x}_i$– nodal coordinates and velocities in undeformed configuration

y_i, $\dot{y}_i$, $\ddot{y}_i$ – nodal coordinates, velocities and accelerations in the deformed configuration

y_{i0}– initial nodal coordinates in the deformed equilibrium state

z_i– nodal displacements from equilibrium position

Δ_r– elastic displacement of the segments; r = 1, 2, 3

ξ – damping ratio

Π_c -complementary configurational potential energy of inextensible cables

ω_{ni} – modal frequencies of vibration; i = 1, 2, 3, 4

Ω – total complementary energy

Ω_c – complementary configurational energy

Ω_e – complementary elastic energy